Numeracy

Outcome 3

[INTERMEDIATE 2]

In Outcome 3, please note that answers for each section are at the end of that section.

INTRODUCTION

The following is an extract from the National Unit Specification of standards.

> ***Outcome 3***
> Apply a wide range of numerical skills.
>
> ***Performance Criteria***
> (a) Work with a numerical concept.
> (b) Decide the operations to be carried out.
> (c) Carry out complex calculations.
> (d) Carry out sustained calculations.

Because this can cover such a wide range of topics, the material for this Outcome has been organised as follows.

First of all there is a **General Section** which covers topics such as basic calculator skills, rounding off to so many decimal places, simple percentages, estimating and checking. Much of this is, frankly, below the level of Intermediate 2, but I think it is best that you at least read it over, even if you don't do the SAQs, to make sure we're on the same wavelength.

There then follow six **Specialised Sections** which cover the following topics:

Applications in Social Arithmetic
More Complex Arithmetical Skills
Ratio and Proportion
Index Numbers
Statistical Calculations
Formulae and Scientific Notation

You will find more details of these on the next page.

The idea is that you won't study them all (well, you can if you're interested) but will either choose, or be directed by your tutor towards, say four of these six sections. Which four will largely depend on your own background and expertise. For instance, if you have a background in business or office work, the first four seem logical; if you were good at, or enjoyed, science subjects when at school, you may find the last two relatively easy.

Discuss the matter with your tutor and make your choice.

Resources
You will only need a calculator for this Outcome. If tackling the Statistical or Scientific Notation sections, a scientific calculator is virtually essential; otherwise a wee cheapie will do.

Assessment

Your tutor will be in touch with you about Assessment. The assessment paper is likely to contain one question on each of the six sections, and you will have to do the four which are relevant to you.

Detailed Contents List for Outcome 3

SECTION 1

1.1 Basic Calculator Skills

The first thing to note is that a calculator will not make a mistake. It takes the numbers which you enter and performs the operations which you tell it to. It spits out the correct answer to the calculation you told it to do. It doesn't know if you meant it to do something else. In particular, it hasn't a clue as to what units you are using.

Example 1.1a

Add together £5.32 and 89p.

Solution:
If you key in 5.32 [+] 89 [=] you will get the answer 94.32 which is wrong, of course.

Either you enter both numbers in £s: 5.32 [+] 0.89 [=] and get the display 6.21 which you must then interpret and write down as £6.21
or you enter both numbers in pence: 532 [+] 89 [=] and get the answer 621 which you must realise means 621 pence, and translate it into pounds and write it down: £6.21.

Example 1.1b

Subtract 394 metres from 2.69 kilometres.

Solution:
Either you enter both in kilometres: 2.69 [−] 0.394 [=] and get the answer 2.296 kilometres
or you enter both in metres: 2690 [−] 394 [=] and get the answer 2296 (metres). It is then up to you to leave it like that, or change it into 2.296 kilometres.

Note that in both cases you key in the 269 number first. The symbol [−] means 'take away' or 'subtract'. It does NOT mean 'from'.

Example 1.1c

Calculate the total cost of 5 bus tickets at 85 pence each.

Solution:
Key in 5 [×] 85 [=] and the display shows 425. Common sense tells you that this actually means £4.25.

You could also key in 85 [×] 5 [=] and get the same answer.

The symbol [×] means 'multiplied by' or 'times'. You get the same answer to a multiplication no matter what the order of multiplying is.

Example 1.1d

Divide a rope of length 3.22 metres into 7 equal lengths.

Solution:
Note first that the symbol [÷] means 'divided by'. It does NOT mean 'into'.

So we must key in the 322 number first, i.e. the one which is being divided. It may be larger OR smaller than the other number, its size does not matter.

Either do it in metres: 3.22 [÷] 7 [=] and the display shows 0.46 which means 0.46 metres or, since you can tell that the answer will be less than a metre, do it in centimetres or millimetres: 322 [÷] 7 [=] 46 cm or 3220 [÷] 7 [=] 460 mm.

The most important thing about using a calculator is that you MUST have the calculation written down on paper before you start pressing any buttons.

A basic, non-scientific calculator
Basic calculators do all their calculations on a left-to-right basis. Every time you press an operator key (+, –, ×, ÷) the calculator calculates what has been put in so far. This may not be actually what you want, so you may have to change the order of doing your calculation.

Example 1.1e

Find the total cost of a cup of coffee at £1.50 and two sandwiches at £1.05.

Solution:
Note that it's quicker doing this one in your head (£3.60), but let's see how a basic calculator will work it out:

If you key in 1.50 [+] 2 [×] 1.05 you will get the answer 3.675 which is plainly wrong. The reason for this wrong answer? As soon as you press the [×] key the calculator adds the 1.50 and the 2, giving 3.50. When you press the [=] key, it will multiply the answer so far (3.5) by 1.05, giving you 3.675.

Knowing this, try keying the calculation the other way round: 2 [×] 1.05 [+] 1.50 [=]
This time the answer is correct. As soon as you press the [+] it calculates what has happened so far, i.e. 2 multiplied by 1.05. Then it adds the price of the coffee.

This 'left-to-right' approach is a basic flaw of this type of calculator, but if you are aware of it you can cope. You may have to use the memory keys on the way, but you'll have to ask the tutor about that; there are too many variations in how they are used to put into print. Better still, read the instructions!

A scientific calculator
Scientific calculators look a lot more complicated because of the vast number of buttons, most of which you won't ever use; however their use of the BOMDAS rule actually makes them easier to operate.

The BOMDAS rule gives you the order of priority which a calculator assigns to any calculation:

Brackets are done before
Of which is done before
Multiplication which is done before
Division which is done before
Addition which is done before
Subtraction

The rule and the examples which follow work for both types of scientific calculator, but if you get into difficulties here, it's best to consult your tutor.

Example 1.1f

Find the total cost of 3 books at £4.99 each and 7 pads of paper at £2.95 each.

Solution:
Key in 3 [×] 4.99 [+] 7 [×] 2.95 [=] and you get the correct answer of £35.62.

This is because the calculator automatically prioritises the two multiplications. As soon as you hit the [+] key, it knows the first multiplication is finished, so it works out the cost of the books. When you hit the second [×] key, it waits to see what will come next. The [=] key tells it the calculation is finished, so it finds the cost of the paper and then brings the addition into use.

(If you did the above calculation in the same order on a basic calculator you would get the answer 64.8115. To get the correct answer you would have to work out the cost of the books, write the answer down, work out the cost of the paper, then add in the books again. Alternatively, work out the books, put the answer into the memory, work out the paper, add the answer into the memory, then recall the memory. Complicated stuff.)

Example 1.1g

Five people go to a restaurant. Only Jim has a starter at £3.95 and he insists on paying that himself. The total of all the main courses is £42.75 and all agree to split the cost of these evenly. What is Jim's share of the bill?

Solution:
You have to divide the 42.75 by 5, and add on the 3.95 so the calculator sequence is

42.75 [÷] 5 [+] 3.95 [=] 12.5 i.e. Jim's share is £12.50

But if you key the calculation in the other way round: 3.95 [+] 42.75 [÷] 5 [=] you get the same answer (only on a scientific calculator) because the calculator will automatically prioritise the division even though you put it in last. There are often distinct advantages in being able to do this.

Example 1.1h

Five people go to a restaurant. The starters come to a total of £18.25; the main courses come to a total of £42.75; the sweets come to a total of £12.50. It is agreed to split the total cost evenly. What is each person's share?

Solution:
You have to add the three amounts of money and divide the answer by 5. This is done in one go by using the brackets.

You would write the calculation like this:

$$\frac{(18.25 + 42.75 + 12.50)}{5}$$

Putting two numbers 'over' each other like this means that you are dividing the top line (called the numerator) BY the bottom line (called the denominator). That is why the division sign (÷) is written like this. The two dots represent the two numbers, one above and the other below the line.

And you key the instructions like this: [(] 18.25 [+] 42.75 [+] 12.50 [)] [÷] 5 [=] and get the correct answer of £14.70.

Important Note

If you key the above sequence without the brackets, like this: 18.25 [+] 42.75 [+] 12.5 [÷] 5 [=] you get the wrong answer of 63.5. Why? Because the calculator will prioritise the division, and thus divide only the 12.5 by 5, then add on the other two numbers. This is why the brackets are so important.

It is fairly important to try to do an entire calculation without any intermediate writing down on paper, because very often mistakes are made when transcribing figures.

It is equally important, as I have said earlier, to write down the calculation before reaching for the calculator. It minimises the risk of making mistakes, and is also essential as evidence of what you did in case it needs to be checked, by your supervisor at work, for example.

If you are new to a calculator you will find the next exercise useful.

If you are already proficient in the use of a calculator you can omit the Exercise and proceed to the page after.

? 1.1

1. A joiner cuts 685 mm off a standard 2.4 m length of timber. How much is left?

2. A typist can type 37 words per minute. How many words per hour is this?

3. A pallett carries 360 bricks. A brickworks produces a batch of 97,200 bricks. How many palletts are needed?

4. A plumber charges £18.60 per hour. How much will he charge for 3 hours' work?

5. A bus arrives at a bus stop with 23 passengers on it. At the stop, 17 more passengers get on but 8 get off. At the next stop 10 get off but 4 get on. How many are on the bus now?

6. Calculate the total cost of 15 reams of paper at £4.53 each and 100 files at £8.20 for each packet of 25.

7. A car travels 14 kilometres on one litre of petrol. How far will it travel on 8½ litres? (You have to enter 8½ as 8.5 on the calculator, unless it has a special fraction key, which can be tricky to use.)

8. A car travels 13 km on one litre of petrol. The tank is virtually empty and the driver has a 585 km drive ahead. How many litres of petrol does he need?

9. £1 is worth 10 French francs. A French tourist intending to visit Scotland has 5,600 francs which he wants to exchange. How many £s will he get?

10. In a chemistry experiment a liquid's temperature is raised to 95°. The liquid then cools at a steady rate of 3° each minute. What is its temperature after 17 minutes?

11. Two couples go to a restaurant for a meal; the cost of everyone's evening is itemised in the table.

	Jim	Donna	Alex	Kirsty
Drinks	£7.45	£3.68	£4.08	£2.79
Starter	£2.99	-	£3.50	£3.95
Main	£8.25	£9.50	£6.99	£11.20
Sweet	£3.49	£2.50	£3.95	-

(a) Which couple spent more, and by how much?

(b) If the total cost for the evening is split evenly, what is the cost to each couple?

(c) If each couple pays for their own drinks, but splits the cost of the food evenly, what is the cost per couple?

1.2 Rounding Off

The answers to the questions in SAQ 1.1 all worked out nicely. But in reality this rarely happens.

Suppose you have two paint suppliers whose prices you want to compare. One sells a 5 litre can of paint for £7.99 and the other a 7.5 litre tin of the same stuff for £10.49. Which offers the better buy ?

You could find the answer in several ways, but one way would be to find the cost per litre for each tin: divide the cost by the number of litres

Tin A: $7.99 \div 5 = 1.598$
Tin B: $10.49 \div 7.5 = 1.3986666667$

Tin A thus costs nearly £1.60 per litre and tin B £1.40 per litre, so tin B is the better buy, all else being equal.

What we have done here is rounded off our calculator display to the nearest penny.

Rounding off is a very important skill and it has to be done properly.

When you round off a number, you **do not change its size**. You simply **show less detail**.

In the next examples we will be rounding off to the nearest whole number, or the nearest ten, or the nearest hundred and so on. We will be rounding to one significant figure, this being shown by the first digit from the left.

Example 1.2a

Round off the number 6.8 to one significant figure.

[This means round it off to the nearest whole number, because the first digit is the 6, which is in the units column.]

Solution:
The number 6.8 means '6 and a bit' and the problem is to decide if the 'bit' is big enough to push the number up to the next whole number (=7), or if the 'bit' isn't big enough and we have to leave it at 6.

Half-way between 6 and 7 is 6.5, and 6.8 is clearly beyond that.

So the answer is 7.

Example 1.2b

Round off the number 83 to one significant figure.

[This means round it off to the nearest ten, because the first digit is the 8, which is in the tens column.]

Solution:
The number 83 means '80 and a bit', so is the 'bit' big enough to push us towards the next ten up (=90), or is it too small and will leave us on 80?

Half-way between 80 and 90 is 85, and 83 is below this half-way point, so the answer is 80.

Example 1.2c

Round off the number 2,346 to one significant figure.

[This means round it off to the nearest thousand, because the first digit is the 2 which is in the thousands column.]

Solution:
The number 2,346 means 'two thousand and a bit'. Half-way between 2,000 and the next thousand up (3,000) is 2,500.

2,346 does not reach 2,500, so the answer remains at 2,000.

? 1.2A

1. Round off these numbers to the nearest whole number:
   (a) 4.7 (b) 6.9 (c) 8.6 (d) 3.12

2. Round these off to the nearest ten:
   (a) 41 (b) 28 (c) 73 (d) 58.7

3. Round these off to the nearest hundred:
   (a) 327 (b) 806 (c) 386 (d) 893.123

4. Round these off to the nearest thousand:
   (a) 3,842 (b) 6,503 (c) 9,670 (d) 14,312

Rounding off to *n* decimal places

Sometimes the answer to a division calculation is exact, and this is always true if you are dividing by 2, 4, 8, 16, etc., or by 5, 10, 20, and by some other numbers. Sometimes we are just lucky!

e.g. $75 \div 4 = 18.75$ exactly $87.3 \div 16 = 5.45625$ exactly

$197.3 \div 5 = 39.46$ exactly $854.34 \div 5.8 = 147.3$ exactly

But usually the answer goes on forever, for instance

$158 \div 13 = 12.1538461$... and this can go on forever

so we would normally round off to a certain number of decimal places.

To count how many decimal places a number has, you count all the digits after the point:

e.g.	12.37	has 2 decimal places
	0.0081	has 4 decimal places
	5.873	has 3 decimal places
	186	has none, since it is a whole number

To round off a number to a number of decimal places we use a similar procedure to the one illustrated in Examples 1.2a–1.2c, here shown in a shortened version.

Example 1.2d

Round off the number 12.692 to 1 decimal place.

Solution:

Step 1 Imagine a line drawn after the first decimal place 12.6|92.

Step 2 The digit after this line is a 9, which is MORE THAN 5.

Step 3 The digit BEFORE this line goes UP 1, making the answer 12.7.

Example 1.2e

Round off 5.3748 to 2 decimal places.

Solution:

Step 1 Imagine a line drawn after the second decimal place 5.37|48.

Step 2 The digit after this line is a 4, which is LESS THAN 5.

Step 3 The digit BEFORE this line stays UNCHANGED, so the answer is 5.37.

Example 1.2f

Look at these and ask yourself why each one is correct.

12.483 to 2 d.p. becomes 12.48
7.2789 to 1 d.p. becomes 7.3
3.85001 to 1 d.p. becomes 3.9

0.0876 to 3 d.p. becomes 0.088
5.699 to 2 d.p. becomes 5.70 (not 5.7 which has 1 d.p.)
11.96 to 1 d.p. becomes 12.0 (not 12)

NB If the digit after 'the imaginary line' is EXACTLY 5 and no more, then the normal procedure is to proceed as if it were in fact more than 5. For example, to round 4.375 to 2 decimal places, since 4.375 is exactly half-way between 4.37 and 4.38 you technically have the choice of both options, but the accepted convention is to go for 4.38.

Example 1.2g

Calculate $\frac{3}{7}$ of 312 metres, and round off the answer to 3 decimal places.

Solution:
The top number of the fraction multiplies, the bottom number of the fraction divides, so key in the following sequence:

3 [×] 312 [÷] 7 [=] and the display shows 133.71429 . . .

The cut-off is between the 4 and the 2. Since 2 is less than 5 the 4 stays. Answer: 133.714 m.

? 1.2B

1. Round these off as required:

(a) 5.832 to 2 d.p. (b) 0.0562 to 2 d.p.

(c) 13.9418 to 3 d.p. (d) 2.9371 to 1 d.p.

(e) 6.9501 to 1 d.p. (f) 0.8749 to 2 d.p.

(g) 187.64 to 0 d.p. (h) 179.6050 to 1 d.p.

(i) 2.3991 to 2 d.p. (j) 0.59999 to 3 d.p.

2. Calculate these, and round off each answer to 3 decimal places:

$\frac{3}{7}$ of £16.23 $\frac{5}{9}$ of 164 km

$\frac{11}{12}$ of 9.74 tonnes $\frac{1}{3}$ of 2.96 litres

1.3 Basic Percentages

The term 'per cent' means **out of one hundred**.

Percentages allow us to compare fractions which are otherwise difficult to compare.

For instance, one company has 123 men and 89 women employed in it, but another has 947 women out of a total workforce of 2,114. Which company employs a greater proportion of women?

It is very difficult to say just by looking at the numbers, but if we convert the figures to percentages we see that the first company has about 42% of its workforce as women and the second has close to 45%. (We'll worry later about how to calculate these figures.)

In other words, if we scale down both companies so that each has a workforce of 100, the first has 42 women (out of the hundred) and the second slightly more at 45.

So, 1% means $\frac{1}{100}$, 5% means $\frac{5}{100}$, 18% means $\frac{18}{100}$, and so on.

Some percentages convert to very easy fractions. We do this by 'cancelling down' the fraction. To do this we divide the numerator and the denominator of the fraction both by the same number. This gives us an **equivalent fraction** which is worth the same but looks different.

Example 1.3a

Express 75% as a common fraction.

Solution:

Put 75 over a hundred. $\frac{75}{100}$

Notice that 5 goes into both numbers so divide each one by 5. $\frac{15}{20}$

Notice that 5 goes into each number again, so divide each by 5. $\frac{3}{4}$

> NOTE:
>
> **Percentage/decimal equivalents**
> 75% means 75/100 which, in decimal form, is written 0.75
>
> Similarly, 24% = 0.24
> 13% = 0.13
> 8% = 0.08
> 1% = 0.01
> and so on.

We have reached the simplest fraction equivalent.

Here are the most common percentages and their fraction equivalents. Many people know these off by heart.

THE HALVES

$50\% = \frac{1}{2}$

THE THIRDS

$33\frac{1}{3}\% = \frac{1}{3}$ $66\frac{2}{3}\% = \frac{2}{3}$

THE QUARTERS

$25\% = \frac{1}{4}$ $75\% = \frac{3}{4}$

THE FIFTHS

$20\% = \frac{1}{5}$ $40\% = \frac{2}{5}$ $60\% = \frac{3}{5}$ $80\% = \frac{4}{5}$

THE EIGHTHS

$12\frac{1}{2}\% = \frac{1}{8}$ $37\frac{1}{2}\% = \frac{3}{8}$ $62\frac{1}{2}\% = \frac{5}{8}$ $87\frac{1}{2}\% = \frac{7}{8}$

THE TENTHS (selection)

$10\% = \frac{1}{10}$ $30\% = \frac{3}{10}$ $70\% = \frac{7}{10}$ $90\% = \frac{9}{10}$

Percentage of something using a calculator

All basic calculators and most scientific calculators have a percentage [%] key. The more advanced scientifics do not!

BUT NOTE: On some calculators the % symbol is marked on the key itself, i.e. you hit that key to get it. But on others it is written (in a different colour) above another key, very often the ‘equals’ key. In this case you have to press the ‘shift’ or ‘2nd’ key before the [%] key.

Ask your tutor if you have problems.

Example 1.3b

Calculate 43% of £154.

Solution if you have a [%] key:

For some reason you usually have to key the quantity first, then the percentage:

154 [×] 43 [%] and it is then a matter of luck if you have to press the [=] key or not.

NB Fractions in percentages:
Some examples:

Key in 12½% as 12.5%
Key in 6¼% as 6.25%
Key in 19¾% as 19.75%
Key in 33⅓% as 33.3333333%
Key in 66⅔% as 66.6666666%
Key in ½% as 0.5%

Try out the sequence above and see. If the answer 66.22 appears right away, fine. If it doesn't, press the [=] key to get it.

Solution if you don't have a [%] key:

You have to key in the 43% as either 43/100 or as 0.43. So your sequence is either

43 [÷] 100 [×] 154 [=] and the display shows 66.22, or 0.43 [×] 154 [=] for the same answer.

Example 1.3c: Increasing by a percentage

A shopkeeper buys some goods for £95 and wants to mark up this amount by 15%. At what price should the goods be sold?

Solution:
On most calculators one of the following two key sequences will work. Just try them in turn to see which one works on yours. (It could be that both or none will work.)

Try either 95 [×] 15 [%] [+] which may give the answer 109.25 meaning £109.25

or 95 [+] 15 [%] which may give the same answer.

If both of these methods fail, you have to do two separate calculations.

First, find 15% of 95, which is 0.15×95, which is 14.25.
Then do a separate addition: 14.25 [+] 95 [=] and obtain 109.25 as before.

Example 1.3d: Decreasing by a percentage

Following a new contour survey, a mountain which has been previously shown on maps as having a height of 2,806 metres must have its height reduced by 1.5%. What will its height be on the new maps?

Solution:
Try either 2806 [×] 1.5 [%] [−] or 2806 [−] 1.5 [%] to give the correct answer 2764 (metres) (rounded off to the nearest whole metre).

If this doesn't work, find 1.5% of 2,806 which is 42.09 and then do 2806 [−] 42.09 [=] and get the same answer.

? 1.3

1. 12½% of a batch of 19,960 bottles produced in a bottle plant are faulty and have to be reprocessed. How many are faulty and how many are not?

2. A safety-conscious employer wants 25% of his workforce to have a first aid certificate. Out of 748 employees, 97 already possess such a certificate. How many more need to obtain one to meet the target?

3. Sally moved up several grades at her work and found that her pay increased by 33$^{1}/_{3}$% as a result. She used to earn £5.85 per hour. How much will she earn per hour now, and what will be her earnings for a 37½ hour week?

4. 87½% of the patients in a hospital one week are women. Out of a total of 1,320 patients, how many are women and how many are men?

5. In a sale, an article previously marked at £130 is reduced by 15%. What is its discounted price?

6. A nurse buys a car for £2,540 and sells it a year later, making a loss of 35%. For how much did she sell it?

7. A dealer bought 100 pairs of training shoes for a total of £1,595 and sold them, making a profit of 140%. What was the selling price of each pair?

8. A wholesaler pays £200 for a TV set, marks it up by 40% and sells it to a retailer. The retailer then marks his cost price up by 30% and sells the set on to a customer. What does the customer pay for it? (And the answer isn't £340 !)

9. A new car cost £12,500 to buy. After one year it had lost 23% of its value. What was it now worth?

10. A speculating company bought a patch of land for £15,000. A few years later it sold the land off, making a profit of 500%. For how much did it sell the land?

1.4 Estimating and Checking

It is amazing how many people accept the calculator answer to a problem as being automatically correct without having a rough idea of the sort of answer they should be expecting. This is one of the main criticisms that employers have of school leavers.

When we estimate the answer to a calculation we do a VERY QUICK calculation, using simplified numbers, which should be so easy to do that we can do it in our head, in order to get a VERY ROUGH IDEA as to what the answer will be.

That way, if we key the numbers into our calculator and end up a mile away from our estimate, we will know that something is wrong.

For example, suppose you have to calculate the total cost of 187 books at £11.95 each.

A rough idea of the correct answer could be to calculate the cost of 200 books at £12 each, which you could do in your head and get an answer of £2,400. You can get this answer in less time than it takes you to pull your calculator out of your pocket.

The correct answer to the calculation 187×11.95 is £2,234.65 and your estimate is quite close to this.

Then, if you key in the actual calculation and get a calculator display of 22346.5 which is about ten times larger than your estimate, you would know that you had keyed something in wrongly. In fact, you put the point in the wrong place.

Of course, estimating won't stop you making silly mistakes like we all do, for instance keying the digits in the wrong order. For example, $178 \times 11.95 =$ £2,127.10 is also close to the estimate, and at first sight you might accept that as the correct answer. But that's where CHECKING comes in.

When we estimate a number we usually do so to one figure accuracy (you'll remember this from page 10 as 'one significant figure').

If the number you wish to estimate is below 1 then you estimate it to the first non-zero digit. The rules you follow are very much like the rules for rounding off.

For example, to estimate 0.0418 look at the first non-zero digit. It is the 4. The next digit is a 1 which is below 5. So your estimate is 0.04.

Now consider the number 0.0007601 in which the first non-zero digit is the 7. Immediately after the 7 is a 6, which is 5 or more. So we will call our estimate 0.0008.

Using estimates in calculations

Example 1.4a: The four rules

Estimate the answer to the following.

(i) 53.2 + 117.8 + 94.032
Solution: 50 + 100 + 100 = 250

(ii) 0.482 + 0.0813 + 0.615
Solution: 0.5 + 0.08 (better 0.1)+ 0.6 = 1.2

(iii) 514.7 – 298.6
Solution: 500 – 300 = 200

(iv) 8.97 × 3.16
Solution: 9 × 3 = 27

(v) 0.89 × 0.214
Solution: 0.9 × 0.2 = 0.18

(vi) 213 ÷ 9.7
Solution: 200 ÷ 10 = 20

(vii) 2,416 ÷ 53
Solution: 2,500 ÷ 50 = 50

In general, you should be able to do an estimate in your head. If you can't, it often means that the numbers you have picked for your estimate are too difficult.

Example 1.4b: Fractions of quantities

Round the fraction to a really easy fraction, and round the number as well.

Estimate these fraction calculations.

(i) 2/3 of 578.1 cm
Solution: 2/3 of 600 = 400 cm
(3 into 600 goes 200 times, then 2 times 200 equals 400)

(ii) 3/7 of £78.24
Solution: 1/2 of £80 = £40

(iii) 7/8 of 2,859 km

Solution: 3/4 of 3,000 = 2,250 km, but you could make it easier by estimating 3/4 of 2,800 because 4 divides into 28 more easily than into 30. This gives an estimated answer of 2,100 km.

There is no 'exact' way to estimate, with no 'correct' or 'wrong' answer. Everyone finds their own method, so long as it makes sense.

Estimating percentages

Round the percentage either to the nearest 10%, or to nice, easy fractions like 1/3 or 3/4.

Example 1.4c

(i) 33% of 142 kg

Solution: either 1/3 of 150 = 50 kg or 30% of 100 kg = 30 kg

(ii) 48% of 2,150 tonnes

Solution: 50% (= 1/2) of 2,000 = 1,000 tonnes

(iii) 23% of £65.20

Solution: 25% (= 1/4) of £64 = £16

Checking your answer

There are various arithmetical ways of checking your answer, but the quickest way is to compare your exact answer with your estimate. If it is miles away, and your estimate is sensible, this suggests that a mistake has been made. If they are close, there is no 100% guarantee of correctness, but most folk are happy with that.

? 1.4

See if you can write down sensible estimates for these. Note that the answers at the back may not tally exactly with yours, but that's OK so long as they are not too far apart.

1.	17.3 + 84.7	2.	26.4 + 163.12	3.	562 + 895
4.	2,356 – 875	5.	0.0512 – 0.0175	6.	9.511 – 2.784
7.	18.3 x 6.9	8.	107.6 x 81.7	9.	243 x 4.9
10.	84.3 ÷ 3.87	11.	123.4 ÷ 18.1	12.	7501 ÷ 4078
13.	$\frac{3}{8}$ of £128	14.	$\frac{5}{9}$ of 71.04 km	15.	$\frac{2}{7}$ of 64.47 kg
16.	9% of 42 pence	17.	42% of £1,892	18.	87% of 512 kg

Answers

?1.1: Answers

1. 1715 mm or 1.715 m
2. 2,220 wpm
3. 270 palletts
4. £55.80
5. 26 passengers
6. £100.75
7. 119 km
8. 45 litres
9. £560
10. 44 degrees

11. (a) Jim £22.18 + Donna £15.68 = £37.86 → £1.40 more
Alec £18.52 + Kirsty £17.94 = £36.46

(b) Total cost split evenly: (37.86 + 36.46)/2 = £37.16

(c) J + D drinks £11.13, food £26.73
A + K drinks £6.87, food £29.59

Food split evenly: (26.73 + 29.59)/2 = £28.16

Final reckoning: J + D: £11.13 + £28.16 = £39.29
A + K: £6.87 + £28.16 = £35.03

?1.2A: Answers

1. (a) 5
(b) 7
(c) 9
(d) 3

2. (a) 40
(b) 30
(c) 70
(d) 60

3. (a) 300
(b) 800
(c) 400
(d) 900

4. (a) 4,000
(b) 7,000
(c) 10,000
(d) 14,000

? 1.2B: Answers

1. (a) 5.83
(b) 0.06
(c) 13.942
(d) 2.9
(e) 7.0
(f) 0.87
(g) 188
(h) 179.6
(i) 2.40
(j) 0.600

2. (a) 6.95571 .... → £6.956
(b) 91.1111 .... → 91.111 km
(c) 8.92833 .... → 8.928 t
(d) 0.98666 .... → 0.987 litres

? 1.3: Answers

1. 2,495 faulty and 17,465 OK
2. 187 need certificates, so 90 to go
3. £7.80 per hour → £292.50 per week
4. 1,155 women → 165 men
5. £19.50 discount → £110.50 marked price
6. £889 loss, so sells for £1,651
7. Profit £2,233 → selling price £3,828 → £38.28 each
8. £200 → £280 → £364
9. Loses £2,875 so worth £9,625
10. Profit £75k, sold for £90k

? 1.4: Answers

NB These are not the only 'correct' estimates (if there is such a thing). Other legitimate variations occur, but they won't be a million miles away from the answers below.

1. $20 + 80 = 100$
2. $30 + 160 = 190$
3. $600 + 900 = 1{,}500$
4. $2000 - 900 = 1{,}100$
5. $0.05 - 0.01 = 0.04$
6. $10 - 3 = 7$
7. $20 \times 7 = 140$
8. $100 \times 80 = 8{,}000$
9. $200 \times 5 = 1{,}000$
10. $80 \div 4 = 20$
11. $120 \div 20 = 6$
12. $8000 \div 4000 = 2$
13. 1/2 of 120 = £60
14. 1/2 of 70 = 35 km
15. 1/4 of 60 = 15 kg
16. 1/10 of 40 = 4p
17. 40% of 2000 = £800
18. 3/4 of 500 = 375 kg

2.1 Buying a House

Before you can get a loan to buy a house the building society or bank from whom you want the loan will send along an inspector to value the property. The fee for such a valuation often depends on the value of the property.

For example, the fees may be set out as follows:

Valuation	Fee
Up to £50,000	Flat fee of £42 + VAT
Over £50,000	£42 plus £1 for every £1,000 or part of £1,000 in excess of £50,000 + VAT

Example 2.1

Find the cost of the valuation fee for a house valued at £74,300.

Solution:
Because of the way the fee is charged ('... or part of £1,000'), you're as well rounding the price up to the nearest £1,000 before you start, so call it £75,000.

The flat fee is £42 for the first £50,000 of value.

There is still £25,000 to pay for, at the rate of £1 per £1,000.
How many thousands is £25,000?
You divide 25,000 by 1,000 and the answer you get is 25, and 25 × £1 = £25.

So the total fee so far is £42 + £25 = £67.

This does not yet include VAT at 17.5% , which comes to £11.73.
So the total amount payable is £67 + £11.73 = £78.73

? 2.1

Calculate the total fees (including VAT) payable on properties with the following valuations:

1. £48,000
2. £60,000
3. £72,400
4. £86,300

2.2 Getting a Loan

A bank or building society won't give you a 100% loan. They expect you to pay a deposit yourself, and will lend you the balance you still require.

Example 2.2

A couple want to buy a house for which the asking price is £74,000. The bank values the house at only £72,000 and is prepared to loan only 90% of this. How much must the couple find as a deposit?

Solution:
The loan is 90% of the valuation, i.e. 90% of £72,000.

$0.90 \times £72{,}000 = £64{,}800$

But the asking price is £74,000.

So the deposit is Asking Price – Loan = £74,000 – £64,800 = £9,200

? 2.2

Calculate the deposit required in each instance below:

	Valuation	Asking Price	Loan %
1	£34,500	£34,500	95%
2	£53,800	£56,200	90%
3	£126,000	£123,250	85%
4	£78,000	£82,000	95%
5	£64,800	£67,300	90%

2.3 Loan Repayment

A **mortgage** is the special name for a loan which is used to buy a house. When you repay a mortgage, the payments are usually made once a month, and you usually repay part of the loan as well as part of the interest owed. Other types of mortgage exist but they are too specialised for this module.

The table below shows what the monthly repayments are, per £1,000 borrowed, for mortgages of different lengths of time at different percentage rates.

Interest rate	10 years	15 years	20 years	25 years
5%	10.79	8.03	6.69	5.91
6%	11.32	8.58	7.27	6.52
7%	11.86	9.15	7.87	7.15
8%	12.42	9.74	8.49	7.81
9%	12.99	10.34	9.13	8.48
10%	13.56	10.96	9.79	9.18
11%	14.15	11.59	10.46	9.90
12%	14.75	12.24	11.16	10.62
13%	15.36	12.90	11.86	11.37
14%	15.98	13.57	12.58	12.12

Example 2.3

Calculate the total amount repaid if £52,340 is borrowed over 20 years at 8%.

Solution:
The table gives the monthly repayment for each £1,000.

Now, £52,340 is 52.34 thousand, and the table entry is 8.49 (highlighted on the table).

The £8.49 is a monthly repayment, so we multiply it by 12 to give the annual repayment for £1,000 and then the answer by 52.34 and finally by 20 (years) to find the grand total:

£8.49 × 12 × 52.34 = £5,332.40 per year

then £5,332.40 × 20 = £106,648 of which £54,308 is just interest!

? 2.3

Calculate the amount actually paid each month to the bank or building society for the following mortgages, and hence the total amount repaid in each case.

1. £45,000 at 10% over 25 years
2. £64,000 at 14% over 15 years
3. £57,800 at 7% over 20 years
4. £35,300 at 12% over 25 years

2.4 House Insurance

There are usually two policies to be considered: contents insurance and buildings insurance. The amount paid annually for this is called the **premium**.

Buildings insurance is often calculated by considering the area of the house.

Example 2.4a

A house is in the shape of a rectangle 10.2 metres by 11.3 metres and is on two floors. The rate for buildings insurance is £0.43 per square metre. Calculate the premium.

Solution:

The premium = Area in square metres $\times$ Rate per square metre
= 10.2 $\times$ 11.3 $\times$ 2 $\times$ £0.43
= £99.12

Contents insurance is often calculated as a rate per cent, i.e. so much per £100 of contents insured.

Example 2.4b

A couple wish to insure the contents of their house for £23,400. What is their premium if the rate is 65p per £100 insured? Normally this has to be paid in one instalment, but for an interest charge of 10% they can pay monthly. What is the monthly payment?

Solution:

£23,400 is the same as 234 hundreds, so the annual premium is
234 $\times$ £0.65 = £152.10

For monthly payments they will have to pay a total of 1.10 $\times$ £152.10 = £167.31 which divides by 12 to give £13.95 per month.

? 2.4

1. Calculate the buildings insurance premiums for the following rectangular dwellings:

   (a) 10.4 m by 13.6 m on one floor, rate 39p per metre^2
   (b) 12.7 m by 9.6 m on two floors, rate 24p per metre^2
   (c) ground floor 15 m by 12.7 m, first floor 12.6 m by 9.3 m, rate 45p per metre^2
   (d) ground floor 22 m by 18.7 m, first floor 18.5 m by 9.9 m, rate 52p per metre^2

2. Calculate contents insurance premiums for the following:

   (a) £12,000 at 75p per £100 (b) £8,500 at £1.23 per £100
   (c) £18,400 at 62p per £100 (d) £21,900 at 48p per £100

2.5 Life Assurance

You can pay monthly premiums towards different types of life insurance (or assurance) policies. Various types exist and you should perhaps find out about these if you haven't done so already.

Age Next Birthday		Term		
	15 Years	20 years	25 Years	30 Years
20	£4.35	£2.98	£2.17	£1.65
21	£4.35	£2.98	£2.17	£1.66
22	£4.35	£2.98	£2.18	£1.66
23	£4.36	£2.98	£2.18	£1.66
24	£4.36	£2.98	£2.18	£1.67
25	£4.36	£2.99	£2.18	£1.67
26	£4.36	£2.99	£2.18	£1.67
27	£4.36	£2.99	£2.19	£1.68
28	£4.36	£2.99	£2.19	£1.68
29	£4.36	£2.99	£2.20	£1.69
30	£4.37	£3.00	£2.20	£1.69
31	£4.37	£3.00	£2.20	£1.70
32	£4.37	£3.00	£2.21	£1.70

All policies calculate the premiums by using a table similar to the mortgage table.

The table shows the monthly premium required per £1,000 assured.

The sum assured is guaranteed, even if you die having paid only one premium (suicides excluded).

An important feature of all this is that all the interest gained is tax free.

Example 2.5

Find the total amount paid over 25 years by a person aged 28 now who wants £30,000 worth of life assurance.

Solution:
The monthly premium for £1,000 assured is £2.20 (highlighted in the table above). Note that, if he is 28 now, he will be 29 on his next birthday.

Total paid $= 30 \times$ £2.20 $\times 12 \times 25 =$ £19,800 which gives him interest of £10,200. (Subtract the total paid from the guaranteed sum assured. A with-profits policy might net even more.)

? 2.5

Use the table above to find the total paid and hence the interest earned on the following policies:

1. aged 22 now, £25,000 assured, after 30 years
2. aged 27 now, £40,000 assured, after 15 years
3. aged 24 now, £15,000 assured, after 20 years
4. aged 30 now, £50,000 assured, after 25 years

2.6 Stocks and Shares

When you buy shares in a company you can make money in two ways:

(1) you can sell the shares at a higher price than you paid, though you will have to pay commission to the stockbroker (not included in these examples)

(2) at the end of each year the shares will yield a dividend, the size of which will depend on how well the company has performed during the year.

In both cases you have to pay income tax on your profit, but we will ignore that for now.

Example 2.6

You buy 250 shares for 286p each. Each share yields a dividend of 43p. You then sell all the shares for 364p each. Calculate your total profit (before tax) and express this as a percentage of your original outlay.

Solution:
Initial outlay $= 250 \times £2.86 = £715$
Dividend $= 250 \times £0.43 = £107.50$
Selling price $= 250 \times £3.64 = £910$
Total profit $= £910 + £107.50 - £715 = £302.50$

$$\text{Profit as a percentage} = \frac{302.50}{715} \times 100 = 42.3\%$$

? 2.6

1. 300 shares at 450p are bought and each share yields a dividend of 22p. They are then sold for 487p each. Calculate the total profit made as a percentage of the original outlay.

2. 400 shares at 374p are bought and each share yields a dividend of 36p. They are then sold for 406p each. Calculate the total profit made as a percentage of the original outlay.

3. 350 shares at 126p are bought and each share yields a dividend of 14p. They are then sold for 185p each. Calculate the total profit made as a percentage of the original outlay.

4. 500 shares at 312p are bought and each share yields a dividend of 25p. They are then sold for 275p each. Calculate the total profit made as a percentage of the original outlay.

2.7 Gross Pay

People are paid in many different ways: by the hour, by the week, by the month, by piecework, by commission, or by a combination of these.

Note that there are 52 weeks in a year, 7 days in a week, which makes 364 days in a year (not that anyone actually works this), yet we are told there are 365 days in a normal year. Similarly, a month (of which there are 12 in a year) has 28 or 29 or 30 or 31 days, so there are not 4 weeks per month.

Needless to say, as you may know only too well, you don't necessarily get to keep all of your earnings. What you actually get paid in total is your **gross pay**. Once deductions have been taken off you are left with your **nett** or **take-home pay**.

Example 2.7a

Fred earns £4.53 per hour, and he works 52 hours in a week.
(a) How much is this for the week?
(b) What is his equivalent monthly salary?

Solution:
(a) Weekly pay is 52 × £4.53 = £235.56

(b) The simplest way to get this in general is to get the equivalent annual salary then divide by 12.

$$\frac{£235.56 \times 52}{12} = £1,020.76$$

Example 2.7b

Angie gets £4.87 per hour for the first 40 hours work during the week, time-and-a-half for any time over this Mon-Fri, and also for Saturday working, and double time on Sundays.

One week Angie works a fair bit of overtime. Altogether she works 43 hours Mon-Fri, 5 hours on Saturday and 4½ hours on Sunday. What is her total gross pay for the week?

Solution:
Basic pay: 40 hrs × £4.87 = £194.80
Time-and-a-half: (3 from the week + 5 from Sat) × (1.5 × £4.87)
= 8 hrs × £7.305 = £58.44
Double time: 4.5 hrs × (2 × £4.87) = 4.5 × £9.74 = £43.83

Total = £194.80 + £58.44 + £43.83 = £297.07

Example 2.7c

A salesman has a basic monthly salary of £500 but he also gets paid commission on sales. The first £25,000 of his sales earns him nothing, but he gets 1% of his next £25,000, 1.5% of the next £25,000 and 2% of any sales over this figure.
Calculate his gross pay in a month when he sells items worth £93,000.

Solution:
We can split his £93,000 as £25,000 + £25,000 + £25,000 + £18,000 with commission
0 1% 1.5% 2%

1% of £25,000	=	£250
1.5% of £25,000	=	£375
2% of £18,000	=	£360
Basic salary	=	£500
TOTAL	=	£1,485

? 2.7

1. Angie gets paid £184.72 per week, Betsy gets paid £831.24 per month.
Who earns more per year?

2. Bert gets a gross pay of £13,203.84 in a year. What is this (a) per week (b) per month?

3. A packer is paid £145 per week basic plus a piecework rate of 2.534p for every item packed. What is her gross pay for a week in which she packs 2,470 items?

4. Nafisha gets paid £4.98 per hour. In addition she is paid a bonus of 3.81 pence for every zip fastener made by her machine. What does she earn during a week when she has worked 38.5 hours and produced 1,956 zip fasteners?

5. Scott works a machine which makes a certain type of component. He gets paid £206.40 per week plus 11.7p for each component completed. During one week he is paid £299.42. How many components did he produce that week?

6. A certain establishment pays £5.82 per hour for a basic 40-hour week, time-and-a-half for anything over this except Sunday work, and double time on Sundays. Calculate the gross pay of Donna who worked 42¼ hours plus 4¾ hours on Sunday.

7. A company pays a rate of £3.1834 per hour, where minutes are expressed as a decimal part of an hour e.g. 36 mins = 36/60 hour = 0.6 hours. Find the gross pay for a shift which started at 14.13 and ended at 21.55.

8. A company pays, in addition to a basic salary, a commission of 1% of the first £20,000 of sales, 1.5% of the next £30,000 of sales, and 2% on sales above £50,000. Find the commission earned by someone who sold (a) £35,000 worth (b) £62,000 worth.

2.8 Deductions

And now the bad news. From your gross pay your employer makes deductions. The most common of these are income tax, national insurance contribution and superannuation.

The rules, entitlements, percentages and amounts for these vary from year to year, so the following exercise will use a 'mickey mouse' collection of information, but the procedure in 'real life' is exactly the same.

Allowances is the name given to money on which the government does not demand tax.

Thus GROSS INCOME – ALLOWANCES = TAXABLE INCOME

The allowances which we will have in this exercise are single person allowance £3,570 (married person's allowance was abolished in the year 2000) and superannuation (here 6.5% of gross income), which is counted as an allowance. It is also, obviously, a deduction.

Tax is counted as a percentage of taxable income. In this exercise we will have

Basic rate:	20% of the first £3,000, then 30% up to £40,000 of taxable income
Higher rate:	40% on taxable income above £40,000

There will still be National Insurance contributions to be paid, but we will not include them here.

Example 2.8

Fred earns £1,250 per month. He is entitled to a single person's allowance. He pays into a superannuation scheme. How much does he keep of his gross pay?

Solution:
Note that all the calculations are done on an annual basis – its too difficult to do otherwise!

Gross annual income = 12 × £1,250 = £15,000

Superannuation = 6.5% of £15,000 = £975

Allowances = £3,570 + £975 = £4,545

Taxable income = £15,000 – £4,545 = £10,455

Tax paid = 20% of £3,000 + 30% of £7,455 = £2,836.50

Annual take-home pay	=	Gross	– Superannuation	– Tax
	=	£15,000	– £975	– £2,836.50
	=	£11,188.50		

and dividing this by 12 gives us the monthly nett pay of £932.375, say £932.38.

? 2.8

Use the figures for tax rates etc. as on the previous page.

And before you complain, I am aware that this is a grossly over-simplified version of reality, but it will save me having to change the numbers after every Budget! In any case, the intention is to familiarise you with the types of calculations, not to turn you into a tax expert!

NB Tax is calculated on complete £s only. So if your taxable income is, say, £5,642.75 the percentages of tax are calculated only on the £5,642.

1. Calculate annual nett pay for a person earning £10,000 per year, single allowance, no superannuation.

2. Calculate monthly nett pay for person earning £1,670 per month, single allowance, with superannuation.

3. Calculate weekly take-home pay for single person earning £342 per week, no superannuation.

4. Calculate monthly take-home pay for married person earning £2,670 per month with superannuation.

2.9 Foreign Currency

When abroad you have to be able to change £s sterling into the foreign currency and vice versa.

The examples and exercise which follow all use the following rates, which were all correct once upon a time.

The quantity of each foreign currency is equivalent to £1.

COUNTRY	RATE for £1	CURRENCY
Austria	**16.85**	schillings
Belgium	**49.35**	francs
France	**8.10**	francs
Germany	**2.39**	marks
Italy	**2,383.21**	lire
Spain	**201.56**	pesetas
United States	**1.56**	dollars

Example 2.9a

A holidaymaker going to Spain exchanged £500 into pesetas, spent 93,250 pesetas, and exchanged the rest back into sterling on his return home. How much did he get back?

Solution:

$£500 \times 201.56 = 100{,}780$ pesetas

$100{,}780 - 93{,}250 = 7{,}530$ pesetas left unspent

$$\frac{7{,}530}{201.56} = £37.35$$

Notice that

- to change sterling into the foreign currency, multiply by the exchange rate
- to change the foreign currency into sterling, divide by the exchange rate.

But life isn't always that simple. For a start, the bank, hotel or exchange office will charge you anything between 1% and 5% for the service (not included in these examples). Also, the bank back home will only change notes, not coins.

Say you have $15.27 to change into £s, you will only be able to change the $15 which (presumably) are in notes. You will be stuck with the 27 cents until your next visit.

This is because banks keep plentiful supplies of notes but not coins.

Example 2.9b

A prospective traveller wishes to change £450 into French francs, then finds she has 127.36 francs left which she wants to change back into sterling. The bank only operates in units of 5 francs. How much money will she get back in sterling?

Solution:
£450 x 8.10 fr = 3,645 francs to spend

She has 127.36 francs left, but since the bank only accepts multiples of 5 francs, she has to keep 2.36 francs as a memento of her holiday and offer 125 francs for exchange.

$$\frac{125}{8.10} = £15.43$$

Example 2.9c

A traveller has 2,340 French francs and enters Germany. Ignoring any commission or problems with notes etc., how many German marks will he receive?

Solution:
8.10 francs are equivalent to £1, and so are 2.39 marks, so we use proportion:

$$\begin{aligned}
8.10 \text{ francs} &= 2.39 \text{ marks} \\
1 \text{ franc} &= \frac{2.39}{8.10} \text{ marks} \\
2{,}340 \text{ francs} &= \frac{2.39}{8.10} \times 2{,}340 \text{ marks} \\
&= 690.44 \text{ marks}
\end{aligned}$$

If you have forgotten how to do proportion, at each stage of the calculation ask yourself the question ‘more or less?’.

If the answer is ‘more’ you multiply.
If the answer is ‘less’ you divide.

? 2.9

1. Convert the following as required using the exchange rates in the table:

   (a) £34 into German marks
   (b) £123 into Spanish pesetas
   (c) $2,340 into £s
   (d) 623 French francs into £s
   (e) 400 Austrian schillings into Italian lire
   (f) 2,000 Belgian francs into French francs

2. A property speculator changed £50,000 into dollars when the exchange rate was £1 = $1.5563. During the next few days the dollar fell by 5.62 cents (i.e. £1 was now equivalent to 1.5563 – 0.0562 = $1.5001) and he changed his money back into sterling. How much profit did he make on the deal?

3. Another speculator changes £2 million into French francs when the exchange rate stands at 8.10 francs to the £. The value of the franc falls by 7 centimes, so he changes it back to sterling. How much of a profit is made?

Answers

?2.1: Answers

1. £49.35
2. £61.10
3. £76.38
4. £92.83

?2.2: Answers

1. £1,725
2. £7,780
3. £16,150
4. £7,900
5. £8,980

?2.3: Answers

1. 45 x £9.18 = £413.10 per month; total £123,930
2. £868.48 per month, total £156,326
3. £454.89 per month, total £109,173
4. £374.89 per month, total £112,467

?2.4: Answers

1. (a) £55.17
   (b) £58.53
   (c) £138.46
   (d) £309.17

2. (a) £90
   (b) £104.55
   (c) £114.08
   (d) £105.12

?2.5: Answers

1. 25 x £1.66 x 12 x 30 = £14,940 giving £10,060 in interest
2. 40 x £4.36 x 12 x 15 = £31,392 giving £8,608 in interest
3. Total paid £10,764 giving £4,236 interest
4. Total paid £33,000 giving £17,000 interest

?2.6: Answers

1. Bought for £1,350; Dividend £66; Sold for £1,461
   Total profit = £177; as a percentage 13.1%
2. Bought for £1,496; Dividend £144; Sold for £1,624
   Total profit = £272; as a percentage 18.2%
3. Profit 57.9%
4. This makes a loss of £60 which is 3.8% of original outlay. Not all stock transactions make a profit. If we had included fees of 1–2% of the value of the shares each time, profits would have been much less. Then there's possible capital gains tax ...

?2.7: Answers

1. Betsy: A £9,605.44 B £9,974.88
2. (a) £253.92
   (b) £1,100.32
3. £207.59
4. £266.25
5. 795 components
6. £307.73
7. £24.51
8. (a) £425
   (b) £890

?2.8: Answers

1. Taxable income = £10,000 – £3,570 = £6,430
   Tax = 20% of £3,000 plus 30% of £3,430 comes to £1,629
   Nett pay = £10,000 – £1,629 = £8,371

2. £1670 per month = £20,040 per year
   Superannuation = 6.5% of £20,040 = £1,302.60
   Taxable income = £20,040 – £1,302.60 – £3,570 = £15,167 (complete £s)
   Tax = 20% of £3,000 + 30% of £12,167 comes to £4,250.10

Nett pay = £20,040 – £1,302.60 – £4,250.10 = £14,487.30 p.a.
Monthly take-home pay = £1,207.28

3. Tax £3,964.20; Nett £265.77 per week

4. Superann. £2,082.60; Tax £7,616.10
Nett £1,861.78 monthly

? 2.9: Answers

1. (a) 81.26 marks
   (b) 24,791.88 pesetas
   (c) £1,500
   (d) £76.91
   (e) 56,574.71 lire
   (f) 328.26 francs
2. £51,873 total, £1,873 profit
3. £17,434 profit

3.1 Hire Purchase

This is a very common method of buying on credit. You pay a deposit followed by regular monthly payments for a period of time. You don't actually own the item until your last payment. By the end of the payment time you will have paid more than the cash price of the item, but you will have had its use for all that time. The extra that you have paid is called the **interest**.

Example 3.1

A television set has a cash price of £429.99, but it can be bought for a deposit of 15% followed by equal monthly payments over two years of £16.99.
Calculate how much more has been paid than the cash price of the TV.

Solution:
Price paid = deposit + monthly payments. Calculate each one separately, then add them.

Deposit = 15% of £429.99 = £64.50

There are 24 (= 2 years × 12 months) monthly payments of £16.99 each. So the total of monthly payments = 24 × £16.99 = £407.76

Total paid = £64.50 + £407.76 = £472.26

But the cash price was only £429.99.

This means that £472.26 – £429.99 = £42.27 was paid in interest over the two years.

? 3.1

1. A video recorder costs £499 cash, or it can be obtained by one of the following methods. Evaluate each one.
   (a) Deposit of 10% and equal monthly payments of £14.99 over 3 years
   (b) Deposit of 20% and equal monthly payments of £18.99 over 2 years
   (c) Deposit of 15% and equal weekly payments of £4.80 over 2 years
   (d) No deposit, but an interest of 16% of the cash price spread over 3 years

2. A dining table costs £699.50 cash, or on credit with no deposit and payments over 18 months. If the total interest charged is 15% of the cash price, what is each monthly payment ?

3. A hi-fi system costs £699 cash. An alternative method of payment is a deposit of 12.5% followed by 11 monthly payments each of 9% of the cash price. How much do you save by paying cash?

3.2 Percentage Increase and Decrease

Many calculators will automatically increase or decrease by a percentage but CARE MUST BE TAKEN that you know how your calculator works.

Example 3.2a

A company finds it must increase the prices of all its goods by 8%. What is the new price of an item previously costing £18.75?

Solution:
Basically, the increase is £18.75 [×] 8 [%] (answer is £1.50)

Then, £18.75 [+] £1.50 [=] £20.25

However, the following sequence works on many calculators:

£18.75 [×] 8 [%] [+] (answer is immediately £20.25)

and others will do this:

£18.75 [+] 8 [%] (answer is immediately £20.25)

Problems arise, however, if you want to calculate, say 10% of £50, but add the answer to £60.

The sequence 50 [×] 10 [%] [+] 60 [=] can very often give the wrong answer (the correct answer is £65).

So be careful. You must know your own calculator!

The trouble with all the above methods, apart from the first one, is that in situations where you want the actual increase, i.e. the £1.50, as a separate quantity (which you might want if it is VAT that you're calculating), you don't get it.

Example 3.2b

A furniture shop offers all its beds at a discount of 12% during a sale. How much would be paid for a bed normally retailing at £529.99?

Solution:
529.99 [×] 12 [%] [−] can give the answer of £466.39

Alternatively, 529.99 [−] 12 [%] on some calculators gives the same answer.

Another way to calculate percentage increases or decreases is to think of the original cost as being 100%, then apply the change. This is the way accountants do it!

Example 3.2c

To increase by	Multiply by
8%	1.08
12%	1.12
17.5%	1.175
75%	1.75
100%	2
200%	3

Increase £8.72 by 12%.

Solution:
If you think of £8.72 as 100%, the new price will be 112%, so you multiply the £8.72 by 1.12 and get £9.77.

Example 3.2d

To decrease by	Multiply by
8%	0.92
12%	0.88
17.5%	0.825
75%	0.25
6.25%	0.9375
32.5%	0.675

Decrease £15.12 by 8.6%.

Solution:
If you think of the £15.12 as 100%, the new amount is 8.6% down on this, i.e. 91.4%, so multiply £15.12 by 0.914 and get £13.82.

? 3.2A

1. A tennis racket used to cost £21.50 but has its price increased by 8%. Find the new price.

2. A holiday company finds that it has to increase the prices of all its holidays by 13%. Find the new cost of holidays which used to cost (a) £450 (b) £730

3. A supermarket drops the prices of some of its goods by 5%. Find the new price of goods which formerly cost (a) £2.99 (b) 79p

4. A football club had an average attendance of 6,243 last season, but this season its performance is down and so is the attendance – by 11%. Find the new average attendance.

5. Fred, Jim and Joe used to earn respectively £312, £350 and £410 per week. Fred gets a 4.4% pay rise, Jim a 3.1% rise, and Joe a 2.6% one. What are their weekly earnings now?

6. Last year a car manufacturer sold 12,400 Albino cars for £9,985 each. In an effort to boost sales, the price was decreased by 8%, and as a result the sales increased by 8%. How did this affect the total value of the sales?

7. A company had a turnover of £2,352,780 last year and required to increase this by 12%. As it turned out, the new turnover was £2,640,150. Was the target met?

Fraction to percentage

There are three types of fractions: vulgar (or common) fractions, decimal fractions and percentages.

For example, these three fractions all say the same thing:

$$\frac{3}{8} = 0.375 = 37.5\%$$

Fraction	Decimal	Percentage
$\frac{3}{5}$	0.60	60%
$\frac{2}{7}$	0.286	28.6%
$\frac{7}{8}$	0.875	87.5%
$\frac{3}{20}$	0.15	15%
$\frac{1}{200}$	0.005	0.5%
$\frac{13}{500}$	0.026	2.6%
$\frac{237}{400}$	0.5925	59.25%

To convert a vulgar fraction into a decimal one, divide the numerator (the top number) by the denominator (the bottom number). Thus

$$\frac{3}{8} = 3 \div 8 = 0.375$$

To convert either of these into a percentage, multiply by 100.

$$\frac{3}{8} \times 100 = 37.5\% \quad \text{or} \quad 0.375 \times 100 = 37.5\%$$

Expressing fractions as percentages helps us to compare them more easily.

Example 3.2e

Fred gets a pay rise of 37.2p per hour on top of his previous rate of £5.32 per hour. Jim, on the other hand, found his monthly salary of £1,280.56 increased by £75.80. Who got the largest increase relative to his original wage?

Solution:

Fred: $\frac{37.2}{532} \times 100 = 6.99\%$

Jim: $\frac{75.80}{1{,}280.56} \times 100 = 5.92\%$

Thus Fred's percentage rise is slightly higher than Jim's.

Note how in Fred's calculation the quantities were both expressed in the same units first, i.e. both in pence. In Jim's case we left them both in £s.

In many cases, as in the example above, the percentage change can be found using the formula

$$\text{Percentage} = \frac{\text{Change}}{\text{Original}} \times 100$$

Example 3.2f

A hi-fi which normally costs £400 cash is bought for a deposit of 15% followed by 11 payments of £42.99 each. Find what the interest rate is over the year.

Solution:

Deposit = 15% of £400 = £60

Payments = 11 × £42.99 = £472.89

Total credit price = £472.99 + £60 = £532.89

Change (i.e. increase) in price = £532.89 – £400 = £132.89

$$\text{Percentage} = \frac{\text{Change}}{\text{Original}} \times 100$$
$$= \frac{132.89}{400} \times 100 = 33.2\%$$

Example 3.2g

Mia earns £915.32 a month and Elizabeth earns £1,065.78 a month. Each girl gets a £35 a month rise. Who got the greater percentage rise and by how much?

Solution:

Mia: $\frac{35}{915.32} \times 100 = 3.82\%$ Elizabeth: $\frac{35}{1,065.78} \times 100 = 3.28\%$

Mia gets 3.82 – 3.28 = 0.54 of a percentage point more than Elizabeth.

? 3.2B

1. In a sale, an item gets reduced from £245.99 to £217.99. Express this reduction as a percentage.

2. In a town in America an item is priced at $35.99 but the customer is, in fact, charged $38.69 because of the local sales tax. What is this tax as a percentage?

3. A shop buys in goods for £12.65 each and sells them for £15.99. What is the shop's mark-up on the goods?

4. A couple bought their house for £65,400 and sold it some time later for £72,600. Solicitor's fees came to 1% of the buying price and 1.5% of the selling price. What percentage profit did the couple make overall?

An accountant would calculate a percentage change in a different way, by using the expression

$$\frac{\text{New}}{\text{Old}}$$

Example 3.2h

A shop buys goods for £32 and sells them for £45. What is the mark-up percentage increase?

Solution:

$$\frac{\text{New}}{\text{Old}} = \frac{45}{32} = 1.40625$$

The '1' represents the 'old £32' and the '.40625' stands for the extra proportion represented by the £13 increase.

If we multiply this decimal bit by 100 we get an increase of 40.625%.

Example 3.2i

A car, originally worth £8,500, is now worth only £7,800. What is its percentage loss in value?

Solution:

$$\frac{\text{New}}{\text{Old}} = \frac{7{,}800}{8{,}500} = 0.9176\ ...$$

It is now worth only 0.918 (rounded off) of what it was before, i.e. 91.8% of what it was before. It has therefore dropped in value by 8.2% (91.8 from 100).

? 3.2C

1. Calculate the mark-up of each of the following as a percentage:

   (a) item bought at £25 and sold at £28
   (b) item bought at £91.42 and sold at £165.99
   (c) item bought at £1,540 and sold at £4,900
   (d) item bought at 53 p and sold at 61p

2. Calculate the depreciation (loss in value) of each of the following as a percentage:

   (a) a drop in price from £10,000 to £8,540
   (b) a drop in price from £816 to £654
   (c) a drop in price from £85 to £34
   (d) a drop in price from £17 to £16

From percentage to original and others
Given that Fred now earns £356 a week after 5% pay rise, what did he earn before? Given that the price of an item is £82.30 but the VAT can be reclaimed, how much can you reclaim?

This section deals with such problems.

Example 3.2j

Fred now earns £356 per week after a 5% pay rise. What did he earn before the raise?

Solution:
If you think of his earning before the raise as 100%, his income after the raise is 105%, which can be also written as 1.05.

So, if the actual amount earned before the raise is X, the amount after the raise is $1.05X$.

Thus

$$1.05X = £356$$
$$X = \frac{£356}{1.05} = £339.05$$

Fred earned £339.05 per week before his raise.

By subtracting £339.05 from £356 you can also find the amount of his actual rise.

Example 3.2k

Given that the price of an item in a shop is £82.30 but the VAT can be reclaimed, how much can you reclaim?

Solution:
If X is the price before VAT, the price after VAT is $1.175X$ because the VAT rate is 17.5%

$$1.175X = £82.30$$
$$X = \frac{82.30}{1.175} = £70.04$$

This is the price before application of VAT. So the actual amount of VAT is

£82.30 – £70.04 = £12.26

The above type of example can be done in a different way:

Example 3.2l

The price after VAT of an item is £175.83. How much of this is the actual tax?

Solution:
As shown in Example 3.2k, the price before tax is

$$\frac{175.83}{1.175}$$

To get the VAT component, subtract this from 175.83.

The calculator sequence is 175.83 [−] 175.83 [÷] 1.175 [=] to which the answer is the VAT component is £26.19.

Note how it is thus completely and utterly WRONG to take the VAT percentage off the final price to find the original price. Watch what happens in the next example.

Example 3.2m

A price of £31.05 includes a VAT component of 15%. Find the original price.

Solution:
You do NOT calculate 15% of £31.05 and take it off. If you did, the answer you would get is £26.39. If you then added 15% back on to £26.39 you would get £30.35 which is NOT the original amount.

You must follow the procedure and divide £31.05 by 1.15 to get the correct answer £27.

If you want the VAT component, subtract the £27 from £31.05 to get £4.05.

You can, if you wish, just learn the rule off by heart:

If VAT is charged at 17.5%, and a price *P* already includes the VAT, to find the price before VAT you divide *P* by 1.175. You can then find the VAT by subtraction. That's the easiest way.

There are other ways taught in text books which deal with this whole affair. Personally, I think this way is the easiest. But stick to the way you know best, even if you didn't read it here.

? 3.2D

1. Assuming a VAT rate of 17.5%, calculate the pre-VAT price of each of these items:
   (a) a TV set retailing at £399.99
   (b) a garage bill which comes to £382.75
   (c) a phone bill which comes to £185.76
   (d) an electric kettle which sells at £21.75

2. Calculate the VAT component of each of the following
   (a) a wardrobe which sells at £250 (VAT rate 17.5%)
   (b) a necklace retailing at $499 (VAT rate 7%)
   (c) a dinner set selling at DM 132.65 (VAT rate 22%)
   (d) a gas bill which comes to £138.75 (VAT rate 8%)

3. A football team announced a 10% increase in its attendance this week compared with last week. If the attendance this week was 9,262, what was it last week?

4. A car costs £11,400 having just undergone a 5% decrease in price. What was its price before?

5. An item in a shop costs £6.96, having just been marked up by 20%. What was its price before the mark-up?

6. Anne got a 6% pay rise and now earns £5.30 per hour. How much did she earn before the pay rise?

7. John now earns £5.86 per hour having just had a pay rise of 8.5%. How much was the actual increase in his hourly rate?

8. A discount of 10% reduces the price of an article to £131.22. What did it cost before the decrease?

9. The local paper reported that the population of a town had increased by 3.2% and now stands at 13,327. What was its population before the increase?

10. A company reported that pre-tax profits had risen to £2,450,000 which represented a 5.6% increase. What had the profits been before?

3.3 Inflation and Depreciation

Inflation means that prices and wages are going up, usually by a percentage. Depreciation means that the prices are going down (usually of goods like cars and equipment), also usually expressed in terms of percentages.

Example 3.3a

A typical 'shopping basket' of goods costs £25.87 in March 1998. In the year to March 1999 the annual rate of inflation was 2.9%, and in the year to March 2000 the rate had further increased by another 0.3 percentage points. In the same time, Mary's weekly wage had gone up from £196.23 to £207.45. How does she stand financially in relation to the shopping basket?

Solution (1):
The inflation to March 1999 is 2.9%, so multiply the price by 1.029.
The inflation to March 2000 is 3.2%, so multiply the 1999 price by 1.032.

So by March 2000 the shopping basket costs £25.87 $\times$ 1.029 $\times$ 1.032 = £27.47

In order to keep pace with inflation, Mary's wage needs to have been increased by the same percentages and should now be £196.23 $\times$ 1.029 $\times$ 1.032 = £208.38

Since she is actually earning less than this, i.e. £207.45, she has fallen behind the inflation rate, so she cannot now buy as much with her money as she could two years earlier.

Solution (2):
The problem can be tackled another way.

The increase in prices over the two years is by a factor of 1.029 $\times$ 1.032 = 1.061928
i.e. an increase of approximately 6.19%.

In the same time, Mary's pay has increased by

$$\frac{\text{Change}}{\text{Original}} = \frac{(207.45 - 196.23)}{196.23} \times 100 = 5.72\%$$

Hence Mary's percentage pay rise over the two years is not as much as the inflation rate over the two years, so she is lagging behind.

Example 3.3b

A car is valued at £23,990 when new. In its first year it depreciates by 20%, then it depreciates by 15% a year for 2 years, and 10% per year thereafter. How much is it worth at the end of four years?

Solution (1):
We can calculate the value of the car year by year:

End of Year 1, down by 20%, now worth 0.80 × £23,990 = £19,192
End of Year 2, down by 15%, now worth 0.85 × £19,192 = £16,313.20
End of Year 3, down by 15%, now worth 0.85 × £16,313.20 = £13,866.22
End of Year 4, down by 10%, now worth 0.90 × £13,866.22 = £12,479.60

Solution (2):
We can go for the final answer in one fell swoop.

At the end of 4 years the car is worth £23,990 x 0.80 x 0.85 x 0.85 x 0.90
= £12,479.60

Sometimes the two methods will give slightly different answers because of the rounding of the figures to the nearest penny. Even rounding the answers to the nearest £ would not really be 'wrong'.

? 3.3

1. A car valued at £35,400 depreciates at the rate of 15% in its first year, 12% in each of the next three years, and 18% per year thereafter. Find its value after
   (a) 3 years (b) 5 years (c) 8 years

2. A painting by Van Gogh cost £3,420,000 in 1994 and appreciated (i.e. increased) in value by 15% per year. What was its value in 1998?

3. A shopping basket of goods cost £45.82. In the three years following, the annual inflation rates are 3.4%, 3.7% and then 3.1%. Find the value of the basket during each of these three years.

4. During the three years of question 3, Andy's monthly wage rose from £1,289 to £1,459. What is his relative financial position?

5. Jenny's wages should be increased by (inflation + 2%) according to a union agreement. At the beginning of the year, a 'shopping basket' was valued at £35.60 and she earned £210 per week. By the end of the year, the price of the basket had been increased by inflation to £36.49. What should her wage increase to?

3.4 Simple Interest

If you have money invested in a bank or building society, your money earns you interest, i.e. you can take out more money than you had put in.

Simple interest is now calculated only when the period of time of the investment is one year or less.

The simple formula $I = \frac{PRT}{100}$ is used to find the amount of interest payable, where

P = Principal = amount of money invested
R = Rate of interest per annum (i.e. annually) in percentage terms
T = Time for which the money is invested, given in years

Example 3.4a

Calculate the amount in the bank if £320 is invested for 85 days at an annual rate of 6.2%.

Solution:

$$I = \frac{PRT}{100} = \frac{320 \times 6.2 \times \frac{85}{365}}{100} \text{ or } 320 \times 0.062 \times \frac{85}{365} = £4.62$$

We assume a normal 365 day year, hence the fraction 85/365

The amount now in the bank is £320 + £4.62 = £324.62.

WARNING
If you try doing the calculation like this: 320 × 1.062 × 85/365 you'll get the wrong answer because the fraction 85/365 is being taken of a different amount. So stick to the method of the example, and no fancy stuff!

Note that, in fact, the situation is not quite as simple as this (as any accountant will tell you) but it's accurate enough for our purposes.

The formula quoted above can be rearranged in terms of any of the other variables:

$$I = \frac{PRT}{100} \qquad P = \frac{100I}{RT} \qquad R = \frac{100I}{PT} \qquad T = \frac{100I}{PR}$$

Example 3.4b

A sum of £450 increased to £474 when the interest rate was 7.3% per annum. For how long had the money been invested?

Solution:
First, the interest is £24 on a principal of £450, so the formula for T gives us

$$T = \frac{100I}{PR} = \frac{100 \times 24}{450 \times 7.3} = 0.7305\ldots \text{ of a year}$$

Multiply this by 365 to get approximately 267 days.

NB For the calculator sequence put both the top line and the bottom line of the calculation into brackets:

(100 x 24) ÷ (450 x 7.3) =

? 3.4

1. Find the interest payable on the following investments:

   (a) £123 for 1 year at 5% p.a. (p.a. means per annum, or 'per year')
   (b) £1,500 for 6 months at 8% p.a.
   (c) £34 for 3 months at 6.4% p.a.
   (d) £100 for 4 months at 5.7% p.a.
   (e) £250 for 8 months at 7.5% p.a.
   (f) £642 for 100 days at 8.2% p.a.

2. A company borrows £12,000 to pay for a new machine. The cost of capital (another way of saying 'the interest charged') is 14%. The loan and interest are to be repaid after one year. How much is repaid?

3. After one year an investor found that a deposit of £346 had increased to £375. What was the rate of interest paid on the deposit?

4. A sum of £56 became £58 after being deposited at 5% p.a. For how long had it been in the account?

5. A company has £2.5 million of capital which it wants to invest for 9 months. The capital is expected to increase to at least £2.75 million in that time. What is the minimum annual interest rate required to produce this return?

3.5 Compound Interest

Again, what follows is a slightly simplified version of the real thing. Interest is added to the principal at the end of each time period and is thus included in the next calculation, just like in the exercises on inflation.

Example 3.5a

A sum of £12,000 is invested in an account which pays 8% p.a. interest, compounded at the end of each year. The account is closed and the money withdrawn at the end of 4 years. How much is realised?

Solution (1):
Strictly speaking, interest is paid on complete £s only, but the amount of difference this makes is very small so we will ignore it.

End of Year 1: 8% of £12,000 = £960, amount in account now £12,960
End of Year 2: 8% of £12,960 = £1,036.80, amount in account now £13,996.80
End of Year 3: 8% of £13,996.80 = £1,119.744, amount in account now £15,116.544
End of Year 4: 8% of £15,116.544 = £1,209.3235, amount in account now £16,325.868

Account closed, amount withdrawn £16,325.87

Solution (2):
The problem is much more quickly done like this

£12,000 $\times$ 1.08 $\times$ 1.08 $\times$ 1.08 $\times$ 1.08 = £16,325.87

or, if you have a power key on your calculator:

£12,000 $\times$ $(1.08)^4$ = £16,325.87

Using the power key:

12000 [×] 1.08 [x^y] 4 [=]
or
12000 [×] 1.08 [^] 4 [=]

Example 3.5b

£400 is invested in an account with an interest rate of 10% p.a. where the interest is compounded quarterly (i.e. every 3 months). How much is in the account after 2 years? How much interest has accrued?

Solution:
We take the 10% p.a. to mean 2.5% per quarter. There are now 8 time periods to consider (four times a year for 2 years)

So amount at end is £400 $\times$ $(1.025)^8$ = £487.36

Actual interest = £487.36 – £400 = £87.36

? 3.5

1. Calculate the total amount in the bank and hence the interest accrued in the following, where the interest is compounded annually.

   (a) £200 at 5% p.a. after 3 years
   (b) £1,900 at 6% p.a. after 4 years
   (c) £375 at 7.5% after 2 years
   (d) £15,000 at 8.3% after 8 years
   (e) A lottery win of £1 million at 10% after 10 years

2. Calculate the total amount in the bank and hence the interest accrued in these, where the interest is compounded at intervals of less than one year.

   (a) £300 at 6% p.a. after 2 years, compounded every 6 months
   (b) £1,000 at 8% p.a. after 2 years, compounded every 3 months
   (c) £4,500 at 4.8% p.a. after 4 years, compounded every 4 months
   (d) £175 at 7.2% p.a. after 3 years, compounded monthly

3. The following compound interest table can be used to calculate the amount after a period of time. Each entry shows the increase in value of **£1** if the interest is compounded annually For other amounts of money, multiply by the amount, e.g. for £246, multiply the appropriate table entry by 246.

Years	5%	6%	7%	8%	9%	10%	11%	12%	13%
1	1.050	1.060	1.070	1.080	1.090	1.100	1.110	1.120	1.130
2	1.103	1.124	1.145	1.166	1.188	1.210	1.232	1.254	1.277
3	1.158	1.191	1.225	1.260	1.295	1.331	1.368	1.405	1.443
4	1.216	1.262	1.311	1.360	1.412	1.464	1.518	1.574	1.630
5	1.276	1.338	1.403	1.469	1.539	1.611	1.685	1.762	1.842
6	1.340	1.419	1.501	1.587	1.677	1.772	1.870	1.974	2.082
7	1.407	1.504	1.606	1.714	1.828	1.949	2.076	2.211	2.353
8	1.477	1.594	1.718	1.851	1.993	2.144	2.305	2.476	2.658
9	1.551	1.689	1.838	1.999	2.172	2.358	2.558	2.773	3.004
10	1.629	1.791	1.967	2.159	2.367	2.594	2.839	3.106	3.395

   Use the table above to find the value of

   (a) £23 at 9% for 8 years
   (b) £145 at 13% for 5 years
   (c) £890 at 10% for 7 years
   (d) £890 at 7% for 10 years

   Are the answers to (c) and (d) the same? Does this surprise you?

3.6 Loan Repayments by Instalments

If you borrow money from a credit company you frequently have to repay part of the loan every so often. Any interest charged is calculated on the reducing balance, i.e. on what you still owe.

Note that this is a very much simplified version of what really happens with credit cards, where you are charged by the day!

Example 3.6

A man borrowed £500 with an agreement to repay £150 in a lump sum at the end of each year. Interest of 8% was to be charged on the amount outstanding each year. How much is outstanding after 3 repayments?

Solution:

End of year 1: owes £500 plus 8% = £500 x 1.08 = £540 but repays £150, so still owes £390

End of year 2 owes £390 plus 8% = £390 x 1.08 = £421.20 but repays £150, so still owes £271.20

End of year 3: owes £271.20 plus 8% = £271.20 x 1.08 = £292.90 but repays £150, so still owes £142.90

? 3.6

1. A person borrowed £2,000 with an agreement to repay £400 in a lump sum at the end of each year. Interest of 8% was to be charged on the amount outstanding each year. How much is outstanding after 3 repayments?

2. A person borrowed £5,000 with an agreement to repay £1,500 in a lump sum at the end of each year. Interest of 10% was to be charged on the amount outstanding each year. How much is outstanding after 3 repayments?

3. A person borrowed £1,000 with an agreement to repay £100 in a lump sum at the end of each month. Interest at the rate of 1% per month is charged on the amount outstanding at the end of each month. How much is outstanding at the beginning of the fourth month?

Answers

? 3.1: Answers

1. (a) £589.54
   (b) £555.56
   (c) £574.05
   (d) £578.84 total (£16.08 monthly)
2. £44.69
3. £779.39; save £80.39 by paying cash

? 3.2A: Answers

1. £21.5 × 1.08 = £23.22
2. (a) £450 × 1.13 = £508.50
   (b) £730 × 1.13 = £824.90
3. (a) £2.99 × 0.95 = £2.84
   (b) 79p × 0.95 = 75p
4. 6,243 × 0.89 = 5,556 people
5. Fred £325.73 Jim £360.85 Joe £410 × 1.026 = £420.66
6. Before change 12,400 cars × £9,985 = £123,814,000
   After change 13,392 cars × £9,186.20 = £123,021,590
   i.e. lost £792,410
7. Target £2,635,113.60 which is just met

? 3.2B: Answers

1. 11.38%
2. 7.5%
3. 26.4%
4. Profit before fees £7,200, but fees of £654 + £1,089 reduce profit to £5,457 which is 8.34% of the original price.

? 3.2C: Answers

1. (a) 12%
   (b) 81.6%
   (c) 218.2%
   (d) 15.1%

2. (a) 14.6%
   (b) 19.9%
   (c) 60%
   (d) 5.9%

? 3.2D: Answers

1. (a) £340.42
   (b) £325.74
   (c) £158.09
   (d) £18.51
2. (a) £37.23
   (b) $32.64
   (c) DM 23.92
   (d) £10.28
3. 8,420 people
4. £12,000
5. £5.80
6. £5
7. 46 pence
8. £145.80
9. 12,914 people
10. £2,320,076

? 3.3: Answers

1. Values of car (to nearest £)

start	£35,400	
end year 1	£30,090	
end year 2	£26,479	
end year 3	£23,302	(a)
end year 4	£20,505	
end year 5	£16,814	(b)
end year 6	£13,788	
end year 7	£11,306	
end year 8	£9,271	(c)

2. $£3,420,000 \times 1.15^4$
   $= £5,981,601$

3. £47.38, £49.13, £50.65

4. Had wage been keeping pace with inflation, his wage would be £1,425, so he's £34 'ahead'.

5. Inflation rate 2.5%
   Pay rise should be 4.5%
   Pay should increase to £219.45

? 3.4: Answers

1. (a) £6.15
   (b) £60
   (c) 54.4p
   (d) £1.90
   (e) £12.50
   (f) £14.42
2. £13,680
3. 8.4%
4. 0.714 of a year = 260 days
5. $13^1/_3$%

? 3.5: Answers

1. (a) £200 x 1.05^3 = £231.52 of which £31.52 is interest
   (b) £1900 x 1.06^4 = £2,398.71 of which £498.71 is interest
   (c) £375 x 1.075^2 = £433.36 of which £58.36 is interest
   (d) £13,386.96 interest
   (e) £1,593,742 interest

2. (a) £337.65 of which interest = £37.65 (4 time periods @ 3% each)
   (b) £1,171.66 of which interest = £171.66 (8 time periods @ 2% each)
   (c) £5,444.24 of which interest = £944.24 (12 time periods @ 1.6% each)
   (d) £217.05 of which interest = £42.05 (36 time periods @ 0.6% each)

3. (a) 23 x 1.993 = £45.84
   (b) 145 x 1.842 = £267.09
   (c) 890 x 1.949 = £1,734.61
   (d) 890 x 1.967 = £1,750.63 which is not the same as (c)

? 3.6: Answers

1. £1,220.86
2. £1,690
3. £727.29

SECTION 4

4.1 Ratio

Ratio is a way of comparing the numbers of one thing with the numbers of a similar thing.

For instance, if a garage sold red cars and blue cars in the ratio 4 : 3 (read 'four to three') then it means that for every 4 red cars sold there were 3 blue cars sold.

e.g.	4 red	:	3 blue
or	8 red	:	6 blue
or	12 red	:	9 blue
or	200 red	:	150 blue

In each case, the proportion of red cars to blue cars remains the same, 4 : 3.

Simplifying ratios
We can simplify ratios, where possible, by dividing both the numbers quoted by another number, thus making the numbers smaller and easier to handle.

Example 4.1a

Simplify the ratio 10 : 6.

Solution:
Think of a number which divides into both 10 and 6. The answer is 2.

So divide each of the numbers 10 and 6 by 2 10 : 6 = 5 : 3

Example 4.1b

Simplify the ratio 1½ hours : 15 minutes.

Solution:
First change both measures into the same unit, in this case minutes:

1½ hours : 15 minutes = 90 mins : 15 mins

Divide both of these by 5 90 : 15 = 18 : 3

Then divide both of these by 3 = 6 : 1

NB When simplifying ratios your answer should contain only whole numbers. You would not, for instance, simplify the ratio 3 : 2 down to 1½ : 1. And some ratios won't simplify at all, for example 13 : 5 is as simple as it gets. If you have three numbers in the ratio, simplify them in the same way, but think of a number which will divide into all three.

? 4.1

Simplify these ratios:

1.	10 : 5	2.	5 : 10	3.	9 : 12	4.	20 : 18
5.	3 : 9	6.	5 : 9	7.	50 cm : 2 m	8.	40 : 35
9.	45 mins : 1 hour	10.	1 kg : 500 g				
11.	15 mm : 25 cm	12.	36 : 72				
13.	3 : 6 : 12	14.	20 : 40 : 25	15.	14 : 10 : 4		

4.2 Calculating with Ratios (1)

Suppose we read that, to make up an orange drink, we must mix orange concentrate with water in the ratio 1 : 4. This means we need 1 measure of concentrate to 4 measures of water (not the other way round, the order of the words and numbers is extremely important). The measures can be spoons, cups or buckets, it doesn't matter so long as they are all the same.

1 measure of concentrate plus **4** measures of water equals **5** measures altogether.

So the concentrate forms $\frac{1}{5}$ of the total and the water forms $\frac{4}{5}$.

Here is another example. A concrete mix requires sand, cement and gravel in the ratio 3 : 5 : 2.

This means we need 3 buckets of sand, 5 buckets of cement and 2 buckets of gravel, in that order, making 10 buckets in all as one 'batch'.

So sand forms $\frac{3}{10}$ of the total,

cement forms $\frac{5}{10}$ of the total

and gravel forms $\frac{2}{10}$ of the total.

As you see, we will be using fractions a lot to solve problems involving ratios. Because ratios are really just fractions under another name.

Example 4.2a

All the cars in a car park are either red or blue. The ratio of red cars to blue cars in this car park is 2 : 3. There are 30 cars in total. How many of each are there?

Solution:
2 measures + 3 measures = 5 measures in all.

So $\frac{2}{5}$ of the cars are red and $\frac{3}{5}$ are blue.

$\frac{2}{5}$ of 30 = 12 red cars and $\frac{3}{5}$ of 30 = 18 blue cars

Check that 12 + 18 = 30 cars in total

Example 4.2b

Bill puts 30p towards a lottery ticket and Ben donates 70p. They win a total of £500. How much does each collect in winnings?

Solution:
Their bets are in the ratio 30 : 70 which simplifies to give 3 : 7.

3 shares + 7 shares = 10 shares in total

Bill gets $\frac{3}{10}$ of £500 which is £150 and Ben gets $\frac{7}{10}$ of £500 which is £350.

Check that £150 + £350 = £500.

? 4.2

1. Cement and sand are mixed in the ratio 2 : 3 to make concrete. How much of each is there in 150 kg of concrete?

2. Sugar and butter are in a cake mix in the ratio 1 : 2. How much of each is there in 900 grams of the mixture?

3. At Tayfirth College the numbers of students and teachers are in the ratio 15 : 1. How many students and how many teachers are there in a group of 3,200 people?

4. A laundry cleans 450 tea towels which are either plain or coloured, the ratio of plain to coloured towels being 5 : 4. If plain ones cost 15p each to clean and the coloured ones 18p each, what is the total cost of laundering the towels?

5. A cake recipe uses 660 g of mixed dried fruit in the ratio raisins : currants : sultanas = 2 : 4 : 5. What are the individual weights?

4.3 Calculating with Ratios (2)

Look at the difference between the last set of examples and this one.

Example 4.3

A concrete mixture contains cement, sand and gravel in the ratio 3 : 4 : 2. There are available 36 buckets of sand, all of which get used. How many buckets of cement and gravel are needed?

Solution:
We do not add the ratio numbers 3, 4 and 2 this time because the 36 buckets refer to the sand only and not to the total weight of the mixture.

So looking at the sand we have

4 shares = 36 buckets, from which it is clear that 1 share = 9 buckets

So cement has 3 shares, which will make 3 x 9 = 27 buckets
and gravel has 2 shares, which will make 2 x 9 = 18 buckets.

? 4.3

1. Silkworm Records sells both compact discs and cassette tapes. The ratio of CDs to tapes sold every day is 3 : 2.
   (a) On Monday they sold 180 tapes. How many CDs did they sell on Monday?
   (b) On Tuesday they sold 120 CDs. How many tapes did they sell on Tuesday?

2. If the ratio of sunny days to wet days is 4 : 7, work out how many sunny days you would expect during a period in which there were 56 wet days.

3. A golfer finds that the ratio of putts to all other shots is 4 : 5 during a round of golf. In this round he had 32 putts. How many other shots were there and what was his total score for the round?

4. In a car park the ratio of red : white : blue cars is 5 : 4 : 1, there being no other colours. If there are 17 blue cars in the car park, and each car is charged £2.50 to park, what is the total amount of money raised?

5. In a group of students the ratio of coffee drinkers to tea drinkers is 5 : 1.
   (a) In one common room there are 10 tea drinkers. How many coffee drinkers are there?
   (b) In another common room there are 35 coffee drinkers. How many tea drinkers are there?

(c) If tea costs 25 pence a cup and coffee 35 pence, and each student has one cup, how much money altogether is raised from the two common rooms ?

4.4 Proportion

If two quantities both increase or both decrease together at the same rate, or in the same ratio, we say that they are in **direct proportion**. For example, if you double one you double the other: 5 boxes cost £7 so 10 boxes of the same type cost £14.

If, as one quantity increases, the other decreases in the same ratio, we say they are in **inverse proportion**. For example, if you double one you halve the other: a 400 watt microwave takes 10 minutes so an 800 watt one will take 5 minutes to cook the same food.

To solve problems involving these we will use the method called the unitary method. (There are other methods too). We won't worry too much about the words direct or inverse, simply whether the number we are looking for is larger or smaller than before.

Example 4.4a

Five books cost £11.25 altogether. Find the total cost of eight books.

Solution:
We start off with the quantity which is mentioned twice. Here, books are mentioned twice, so we start with them.

5 books cost £11.25

Now we find the cost of one book.
Since one book costs less than 5 books, we divide by 5.

1 book costs $\frac{£11.25}{5} = £2.25$

Since 8 books cost more than 1 book we multiply by 8.

8 books cost 8 x £2.25 = £18.00

This was an example of direct proportion:
as the number of books goes up, the total cost goes up
as the number of books goes down, the total cost goes down.

IMPORTANT RULE

If you want MORE you must MULTIPLY.

If you want LESS you must DIVIDE.

Example 4.4b

If a car travels 132 km on 4.5 litres of petrol how far will it travel on 3.7 litres?

Solution:
The petrol is mentioned twice so start with that:

4.5 litres takes the car 132 km

1 litre will take the car less far, so divide: $\frac{132}{4.5}$ but we do NOT need the answer to this yet!

3.7 litres will take the car further than just 1 litre, so multiply by 3.7 and do the whole calculation at one go:

$$\frac{132}{4.5} \times 3.7 = 108.533 \ldots \text{ which rounds off to } 108.53 \text{ km}$$

Notice that the final answer for 3.7 litres is less than the distance for 4.5 litres which is as expected.

YOU SHOULD ALWAYS HAVE A ROUGH IDEA OF WHAT TO EXPECT.

Example 4.4c

A car travelling at 30 miles per hour takes 4 hours to complete a journey. At what speed will the car need to travel to do the journey in only 3 hours ?

Solution:
Time is mentioned twice in the question so start with that:

4 hours is needed at 30 miles per hour

To do the journey in only 1 hour, the car needs to go a lot faster, at 4 x 30 miles per hour, in fact. (Remember the rule, if MORE then MULTIPLY.)

But to do the journey in 3 hours it needs to go more slowly than if in 1 hour (If LESS then DIVIDE):

$\frac{4 \times 30}{3}$ = 40 miles per hour

The whole calculation can be laid out like this:

4 hours $\rightarrow$ 30mph

1 hour $\rightarrow$ (more) 4×30

3 hours $\rightarrow$ (less) $\frac{4 \times 30}{3}$ = 40mph

? 4.4

1. Ten biscuits cost 70 pence. (a) How much will 6 biscuits cost ? (b) How many biscuits can be bought for £1.40?

2. Three tapes last for 75 minutes altogether. How long would you expect 8 such tapes to last?

3. A car travelling at 50 miles per hour does a journey in 6 hours. How fast does the car have to go to do the journey in 3 hours ?

4. A tourist changes £30 for 330 Danish kronor. How many kronor would she get for £20?

5. A camp cook has enough food for 24 Brownies for 5 days. 30 Brownies actually turn up at the camp. For how long will the food last ?

6. It takes 15 men 7 days to do a job. How long would it take 21 men to do the same job, working at the same rate (i.e. assuming they don't get under each others' feet)?

7. Towels cost £29.70 for 6 and sheets cost £35.80 for 4. Find the total cost of 5 dozen towels and 20 sheets.

8. Eight bus passes can be bought for £36.40. How many can be bought for £236.60?

9. A 650 watt microwave oven takes 15 minutes to cook a casserole. How long would a 900 watt oven take to cook the same casserole ? (More powerful microwave cooks faster.)

10. Four bricklayers can finish a job in 20 days but the foreman decides the job must be completed in 8 days instead. How many extra brickies are needed?

11. A 5-metre roll of flex costs £11.95. For a rewiring job in a hospital ward an electrician decides he needs 140 metres of flex. What will be the cost?

12. A famine relief worker estimates that 350 crates of milk powder are needed to feed 5,300 children at a feeding centre. In fact, 39,750 children need fed. How many crates of milk powder are required instead?

13. A bottle of medicine contains enough for 150 doses of 5 ml each. The doctor decides to increase the dose to 7½ ml. How many doses can the nurse get from one bottle now?

14. To fill a garage base of volume 3.6 m^3 costs £125.28. How much does it cost for a base of volume 2.7 m^3?

15. A United Nations relief worker estimates she has enough food to feed 5,000 refugees for a month. In fact, more than 20,000 refugees turn up. For how long will the food last without the delivery of further supplies?

4.5 More Complex Proportion

Up till now, the problems all involved two quantities. More complex situations exist with three or more quantities.

Example 4.5a

Three workers earn a total of £540 in 12 hours. In how many hours will 9 workers earn a total of £1,080?

Solution (1):
We can argue the whole thing out logically. If 3 people, working for 12 hours each (i.e. 36 hours' worth of work), earn a total of £540, then the hourly rate (i.e. the earning for 1 hour's work) is £540 ÷ 36 = £15 per hour.

Next, 9 workers earn a total of £1,080 which means £1,080 ÷ 9 = £120 for each person.

If they are earning £15 per hour, then each must work £120 ÷ £15 = 8 hours.

Solution (2):
Do the whole lot by proportion, worrying only about 'less' or 'more'.

Here, the answer is in hours, so write the statement of information with the hours last:

3 people → £540 → 12 hours
9 people → £1,080 → ? hours

We are now going to multiply by fractions according to the following rules:

> If the answer at any stage is 'less' you multiply by a fraction with the smaller number on the top.
>
> If the answer at any stage is 'more' you multiply by a fraction with the larger number on the top.

It will take 9 people less time to earn an amount of money than it takes 3 people to earn the same, so we multiply by the fraction

$\frac{3}{9}$ (multiplying by '3 over 9' gives a smaller answer than multiplying by '9 over 3')

It takes more time to earn £1,080 than it does to earn £540, so we multiply by the fraction

$\frac{1{,}080}{540}$ (multiplying by '1,080 over 540' gives a larger answer than multiplying by '540 over 1,080').

So we can lay out the whole question like this:

3 people $\rightarrow$ £540 $\rightarrow$ 12 hours

9 people $\rightarrow$ £1,080 $\rightarrow$ $12 \times \frac{3}{9} \times \frac{1{,}080}{540} = 8$ hours

Calculator sequence: 12 [×] 3 [÷] 9 [×] 1,080 [÷] 540 [=]

Example 4.5b

An engineering firm obtains a contract to complete a 1.6 km tunnel in 5 years 8 months. For 3 years there are 1,200 men employed and only 600 m of the tunnel has been completed. How many additional men must be employed in order to complete the work by the date specified in the contract?

Solution:
We write out the information statement with men at the end, since the answer is a certain number of men.

600 m completed in 36 months by 1,200 men

The tunnel is 1,600 m long altogether, so 1,000 metres are left to be built. There are 2 years and 8 months of time left. Notice that it is easier if we change the time to months:
2 years 8 months = 32 months.

$$1{,}000\text{m completed in 32 months by } 1{,}200 \times \frac{1{,}000}{600} \times \frac{36}{32} \text{ men}$$

1000 m takes <u>more</u> men to complete than 600 m so the <u>big</u> number goes on top.

To complete the project in less time requires <u>more</u> men, so the <u>big</u> number goes on top.

The answer is 2,250 men are needed to complete the project on time. If we subtract the 1,200 who are already working, we need an extra 1,050 workers.

Example 4.5c

When a litre of petrol cost 70p a litre, a car which travelled 9.2 miles per litre cost its owner £25 for a particular journey. The price of petrol goes up to 75p a litre, the car has just been serviced, and its consumption improves. It now manages 10.6 miles per litre. How much does the same journey cost?

Solution:

70p per litre →→→ 9.2 miles per litre →→→ £25

75p per litre →→→ 10.6 miles per litre →→→ $25 \times \frac{75}{70} \times \frac{9.2}{10.6} = 23.2479 \ldots$

At 75p per litre the journey will cost <u>more</u> than at 70p, so the <u>bigger</u> number goes on top

The improved consumption means that it will cost <u>less</u>, so the <u>smaller</u> number goes on top

The answer rounds off sensibly to £23.25.

You can work out Example 4.5c using common sense quite easily, but Example 4.5b is a lot harder unless you use the proportion method.

? 4.5

1. 200 men working 12 hours a day can make an embankment 5 km long in 25 days. How many hours a day will 60 men have to work to complete a similar embankment 2 km long in 32 days?

2. 1,250 copies of a book of 352 pages require a total of 55 reams of paper. How many reams of paper are needed for 1,375 copies of a book of 384 pages?

3. A consignment of 288 metal tubes for an American factory, each tube 7 feet 6 inches long, weighs 325.2 pounds. Find the weight of 400 similar tubes, each 4 feet 6 inches long. (America has not yet gone metric – there are 12 inches in 1 foot. And Americans weigh in pounds, not kilograms.)

4. A contractor undertakes to dig a canal 6 miles long in 175 days and employing 45 workers. The contract is completed on time. How long a canal could be dug in 150 days with 50 workers working at the same rate?

5. A railway company contracts to carry goods weighing 3.6 tonnes 320 miles at a charge of £450. What weight of goods would be carried 200 miles for £300?

6. 5 tonnes of rice can provide nourishment for 12,000 refugees for 4 days. In fact, the refugee centre has to cater for 50,000 people and 18 tonnes of rice are delivered. For how long will the rice last?

7. A castle under seige has 1,500 inhabitants and enough food to last 18 days at the rate of 750 grams per day. 300 people manage to break out and the ration for the remainder is reduced to 500 grams per day. For how much longer will the castle hold out?

8. A contractor is engaged to remove 4,275 m^3 of earth in 20 days with a workforce of 30 people. Half-way through the contract it is found that only 1,000 m^3 have been removed. How many extra people must be taken on to complete the work on time?

9. Last year a travelling salesman found he ran up a bill of £1,800 when driving a distance of 16,200 miles in a car whose consumption was 6.5 miles per litre of petrol costing 75 pence per litre. This year his car is more economical – 8 miles per litre, but petrol has increased by 3.5 pence per litre and he drives 20,400 miles. What is the total cost?

10. One for the campers among you! The gas consumed in 25 days by a stove which burns 1.8 m^3 per hour costs £15. For how many hours per day is it lit if the cost of gas for a stove burning 10 hours a day for 28 days and using 1.5 m^3 per hour is £11.67?

Answers

4.1: Answers

1. 2 : 1
2. 1 : 2
3. 3 : 4
4. 10 : 9
5. 1 : 3
6. 5 : 9
7. 1 : 4
8. 8 : 7
9. 3 : 4
10. 2 : 1
11. 3 : 50
12. 1 : 2
13. 1 : 2 : 4
14. 4 : 8 : 5
15. 7 : 5 : 2

4.2: Answers

1. 60 kg cement, 90 kg sand
2. 300 g sugar, 600 g butter
3. 3000 students, 200 teachers
4. 250 plain + 200 coloured = £73.50
5. 120 g raisins, 240 g currants, 300 g sultanas

4.3: Answers

1. (a) 270 CDs
   (b) 80 tapes
2. 32 sunny days
3. 40 others, 72 round
4. 17 blue, 68 white, 85 red = £425
5. (a) 50 coffee
   (b) 7 tea
   (c) 85 coffee + 17 tea = £34

?4.4: Answers

1. (a) 42p
   (b) 20 biscuits
2. 200 minutes
3. 100 mph
4. 220 kroner
5. 4 days
6. 5 days
7. £297 + £179 = £476
8. 52 passes
9. 10.83 minutes = 10 minutes 50 seconds
   (to change 0.83 minutes into seconds, multiply it by 60)
10. 10 brickies = 6 extra
11. £334.60
12. 2,625 crates
13. 100 doses
14. £93.96
15. About 7 or 8 days

?4.5: Answers

1. $12 \times \frac{2}{5} \times \frac{25}{32} \times \frac{200}{60} = 12\frac{1}{2}$ hours per day

2. $55 \times \frac{1{,}375}{1{,}250} \times \frac{384}{352} = 66$ reams

3. $325.2 \times \frac{400}{288} \times \frac{54}{90} = 271$ pounds

4. $6 \times \frac{50}{45} \times \frac{150}{175} = 5.71$ miles

5. $3.6 \times \frac{200}{320} \times \frac{300}{450} = 1.5$ tonnes

6. $4 \times \frac{18}{5} \times \frac{12{,}000}{50{,}000} = 3.456$ days

7. $18 \times \frac{1{,}500}{1{,}200} \times \frac{750}{500} = 33\frac{3}{4}$ days, i.e. $15\frac{3}{4}$ days longer

8. $30 \times \frac{3{,}275}{4{,}275} \times \frac{20}{10} = 45.9$, i.e. 16 more people

9. $1{,}800 \times \frac{204}{162} \times \frac{6.5}{8} \times \frac{78.5}{75} = £1{,}927.61$

10. $10 \times \frac{28}{25} \times \frac{1.5}{1.8} \times \frac{15}{11.67} = 12$ hours per day

SECTION 5

5.1 Simple Price/Quantity Index

An index is a measure, over a period of time, of the average changes in the values (usually either prices or quantities) of a group of items. It expresses the value now (i.e. the current value) as a percentage of the value at some previously fixed time called the **base date**. Usually we deal in annual indices, in which case the base date is the **base year** and is given the index **100**.

A company makes wide use of index numbers to evaluate its trading position in relation to its competitors. It will also rely on national indices (the plural of index) of prices, production, wages, sales, transport charges and share prices to provide a background against which decisions affecting the company can be taken.

Anyone concerned with labour must necessarily be interested in indices of hours of work, earnings, wage rates and retail prices.

Perhaps the most well-known index is the Retail Price Index (RPI) which measures the average level of a 'shopping bag' of approximately 600 of those goods and services which are most widely used by households in Britain. It is frequently used as a basis for national wage negotiations between management and unions. More about this later.

There are many indices in common use, each with its own strengths and weaknesses, and we will study a few of these.

The first is a **Simple Price Relative Index**, the formula for which is:

$$\frac{p_n}{p_o} \times 100$$

Where p_o is the price for the base year and p_n is the price for any other year.

If it helps, you can think of p_o as meaning 'p old', or 'old price', the word old referring to the base year; and p_n as meaning 'p new' or 'new price', the word new referring to whichever year you are calculating an index for.

Example 5.1a: A simple price index or price relative index

The table shows the average price of a family car for a number of years. Calculate a price index with 1992 as the base year.

Year	1992	1994	1996	1998
Price (£)	5,200	5,950	6,200	6,740

Solution:
If 1992 is to be the base year we give it the index number 100.

(Any year can be made the base year, it doesn't have to be the first year in the table.)

We now apply the formula to each of the other years. What we are effectively doing is expressing each price in percentage terms compared with the base year price of 100:

The index for 1994 is $\frac{5{,}950}{5{,}200} \times 100 = 114.4$ showing that it is 14.4% above the 1992 price

The index for 1996 is $\frac{6{,}200}{5{,}200} \times 100 = 119.2$ showing that it is 19.2% above the 1992 price

The index for 1998 is $\frac{6{,}740}{5{,}200} \times 100 = 129.6$ showing that it is 29.6% above the 1992 price

Note again: these index numbers show how the average price relates in percentage terms to the price at the base year.

We can now present the results by adding another row to the table above:

Year	1992	1994	1996	1998
Price (£)	5,200	5,950	6,200	6,740
Index	100	114.4	119.2	129.6

So the index for 1996 may be 4.8 percentage points higher than the index for 1994 (because 119.2 – 114.4 = 4.8) but it is not 4.8% higher! We have to watch our language!

What the table shows is that the 1996 price is 19.2% higher than the 1992 (base year) price.

The 1996 price is actually 4.2% higher than the 1994 price. Can you think what calculation I did to arrive at that figure?

Example 5.1b: A simple quantity index or quantity relative index

A hotel notices that the number of cans of orange juice sold by its bar over a period of time is as follows:

Year	1993	1994	1995	1996	1997
No. of cans	625	684	705	690	745

Calculate a simple quantity index, making 1994 the base year.

Solution:
As I said earlier, the base year does not have to be the first year in the table, it can be any year.

We start by making 1994 have the index 100. For each year we now calculate the fraction

$\frac{q_n}{q_o} \times 100$ where q_n is that quantity in that year, and q_o is the quantity in the base year.

Now calculate:

1993 index is $\frac{625}{684} \times 100 = 91.4$ showing the quantity is 8.6% below that for 1994

1995 index is $\frac{705}{684} \times 100 = 103.1$ showing the quantity is 3.1% above that for 1994

1996 index is $\frac{690}{684} \times 100 = 100.9$ showing the quantity is 0.9% above that for 1994

1997 index is $\frac{745}{684} \times 100 = 108.9$ showing the quantity is 8.9% above that for 1994

? 5.1

Calculate price or quantity indices as appropriate.

1.

Year	1985	1986	1987	1988	1989
Quantity (tons)	418	496	512	554	572

Make 1985 the base year

2.

Year	1987	1988	1989	1990	1991
Sales (£m)	25.7	27.4	29.9	26.3	24.8

Make 1987 the base year

3.

Year	1991	1992	1993	1994	1995
Price (pence)	18	25	28	36	42

Make 1993 the base year

A basic index such as the one studied so far can be applied to a total sales or volume figure. More usually, however, the figure is made up of components of different sizes, and so we apply slightly different techniques.

5.2 More Complex Indices

Example 5.2a: Simple aggregative (unweighted) index

Suppose an ice cream shop sells five varieties of ice cream with prices of a 2-litre tub as shown in this table. The owner is looking for a quick, simple index which will reflect the different types of price changes over the whole range of ice creams.

Type of Ice Cream	July 1993	July 1994
Plain Vanilla	£2.10	£2.10
Harlequin	£2.85	£3.05
Caribbean Fancy	£3.45	£3.90
Choconut	£2.65	£2.60
Mint 'n' Mango	£2.90	£3.20

As you see, some have increased in price (by varying amounts), some have decreased, and one has remained the same. So we are looking for an index which will try to take these differences into account.

The general formula for this type of index is

$$I = \frac{\sum p_n}{\sum p_o} \times 100$$

The $\sum$ sign means 'the sum of'. So the formula is telling us to add up all the p_ns, add up all the p_os, divide the first answer by the second, and then multiply by 100.

Solution:

This is the total of all the prices in the 'new' year 1994

$$\frac{(2.10 + 3.05 + 3.90 + 2.60 + 3.20)}{(2.10 + 2.85 + 3.45 + 2.65 + 2.90)} \times 100 = 106.4516 \quad .....$$

$$= 106.5 \text{ (to 1 decimal place)}$$

This is the total of all the prices in the 'old' (= base) year 1993

Remember to put in the brackets when using the calculator.

Another way to compare the two sets of figures is to calculate a separate price relative for each product, convert to an index, then find the arithmetic mean of all these indices.

The formula is

$$I = \frac{\sum \frac{p_n}{p_o} \times 100}{k}$$

where k is the number of items whose prices are being compared.

Example 5.2b: Arithmetic mean of price relatives (unweighted)

Calculate the arithmetic mean of price relatives index for the data of Example 5.2a.

Solution:

This is the price relative for Harlequin ice cream

This is the price relative for the Choconut flavour

$$\frac{\frac{2.10}{2.10} + \frac{3.05}{2.85} + \frac{3.90}{3.45} + \frac{2.60}{2.65} + \frac{3.20}{2.90}}{5} \times 100 = 105.7038 \ .....$$

$$= 105.7 \ \text{(to 1 dec. place)}$$

As you see, the answer is slightly different. The method is also slightly longer, but it has the advantage of calculating an individual index for each brand of ice cream as well as an average index.

If we replace the ps by qs we obtain similar indices for quantities rather than prices.

? 5.2

Compute (a) the simple aggregative index and (b) the arithmetic mean of relatives index (for prices or quantities as appropriate) for each of the following, using the figures for the earlier year as base.

1. Weekly wages paid to four categories of workers in a factory.

Category	1980	1985
unskilled	£85	£95
semi-skilled	£93	£105
skilled	£110	£135
clerical	£100	£114

2. Average price paid per bottle of wine in a shop.

Type of Wine	1990	1995
Cheap plonk	£1.95	£2.10
Chateau Bottled	£2.89	£3.90
Vintage	£5.60	£6.00
Grand Cru	£12.84	£15.99

3. Number of electrical appliances sold in Outer Monrovia (in thousands).

Appliance Type	1993	1997
Refrigerators	431	442
Freezers	127	196
Dishwashers	86	154
Washing Machines	295	306
Tumble Dryers	104	128

5.3 Paasche's Index

If we take a look back at our ice cream scenario, neither of the two indices we found take into account the quantity of each brand sold. It could be, for instance, that the shop sold 100 times as many tubs of Choconut as of all the rest put together.

Unweighted indices, therefore, are of limited use. All of the more common indices actually used in real life are **weighted indices**.

And what are weights?

Weights are simply numbers which reflect the relative importance of a particular item when compared with other items.

The most common index numbers in general use are both examples of weighted aggregative index numbers, and were invented during the nineteenth century, one by Etienne Laspeyres and the other by Hermann Paasche.

When faced with the choice of using figures from the base year or figures from the current (or new) year as weights, Paasche opted for the current year, arguing that they are of more relevance, and it is Paasche's Index that we will look at now.

(Laspeyres, on the other hand, argued for using the base year figures as weights – an argument we won't go into here!)

If we are calculating an index for prices, we use the quantities as weights.
But if we wish to calculate an index for quantities, we use the prices as weights.

Paasche simply pretended that the figures to be used as weights were the same in the base year as they were in the current year. This is because, if we want an index of prices, there is no point of changing both prices and quantities – we will not then know how much of the index depends on which figure.

For an index of prices we keep the quantities fixed (at current year levels), so that their importance is recognised but at least we know that the change in the index is due entirely to the change in price.

Similarly, for an index of quantities we keep the prices fixed (at current year levels) so that we are assured that the change in the index is due to changes in the quantities.

Example 5.3

Consider the example on ice cream from a previous page.

Type of Ice Cream	July 1993		July 1994	
	Price per Tub p_o	No. of Tubs q_o	Price per Tub p_n	No. of Tubs q_n
Plain Vanilla	£2.10	15	£2.10	18
Harlequin	£2.85	24	£3.05	32
Caribbean Fancy	£3.45	16	£3.90	20
Choconut	£2.65	20	£2.60	19
Mint 'n' Mango	£2.90	30	£3.20	35

Calculate Paasche's index for prices and Paasche's index for quantities sold, for 1994, taking 1993 as base year.

Solution:
The formulae are:

Paasche's Price Index	**Paasche's Quantity Index**
$I = \dfrac{\sum p_n q_n}{\sum p_o q_n} \times 100$	$I = \dfrac{\sum p_n q_n}{\sum p_n q_o} \times 100$

Look at the price index. The price on the top is p_n, the current year price, and the price on the bottom is p_o, the base year price. Each of these prices is being multiplied by q_n, the current year quantity. We are pretending that the quantities sold in 1993 are actually the same as the quantities sold in 1994. In that way the quantities sold are still playing a part, but we are at least sure that the index will be a result of the changes in the prices alone, since the quantities are being held constant.

A similar argument pertains to the quantity index. If it is a quantity index we are calculating, then we must change the quantities, hence q_n on top and q_o on the bottom of the fraction. But we will pretend that the prices in 1993 were the same as they were in 1994. That way we are sure that the change in the index will be due to the changes in quantities, but the prices are still playing a part.

Had we used the formula

$$\frac{\sum p_n q_n}{\sum p_o q_o} \times 100$$

we would see that both prices and quantities are changing, so we wouldn't know how much of the index change was due to changes in price, and how much was due to changes in quantities.

At least, that's the theory.

Back to the calculation. It is customary to lay out the results in tabular form:

We need a column headed $p_n q_n$, meaning $p_n \times q_n$ and we get the entries in it by multiplying each value of p_n by its corresponding q_n value. Thus we get 37.80 by multiplying 2.10 by 18. Notice that we don't have £ signs in front of this! We add up the entire column when finished.

Similarly we need a column for $p_o \times q_n$ and another for $p_n \times q_o$ because these are what the formulae require. For example, we get the entry 69.00 by multiplying 3.45 (p_o) by 20 (q_n).

p_o	q_o	p_n	q_n	$p_n q_n$	$p_o q_n$	$p_n q_o$
£2.10	15	£2.10	18	37.80	37.80	31.50
£2.85	24	£3.05	32	97.60	91.20	73.20
£3.45	16	£3.90	20	78.00	69.00	62.40
£2.65	20	£2.60	19	49.40	50.35	52.00
£2.90	30	£3.20	35	112.00	101.50	96.00
			Totals	374.8	349.85	315.1

The final calculations:

Paasche's Price Index: $I = \frac{\sum p_n q_n}{\sum p_o q_n} \times 100 = \frac{374.8}{349.85} \times 100 = 107.1$

Paasche's Quantity Index: $I = \frac{\sum p_n q_n}{\sum p_n q_o} \times 100 = \frac{374.8}{315.1} \times 100 = 118.9$

In other words, the prices have gone up by an average of 7.1%, and the quantities have gone up by an average of 18.9%, from 1993 to 1994.

Now for some practice.

? 5.3

1. Here are the wage rates and the number of employees for four categories of worker.

	1993		1995	
Category	Wage	No. of Employees	Wage	No. of Employees
unskilled	£85	10	£95	15
semi-skilled	£93	25	£105	35
skilled	£110	45	£135	60
clerical	£100	1	£114	2

Calculate
(a) Paasche's index for prices and
(b) Paasche's index for employee numbers, both for 1995, taking 1993 as base.

2. Here are details of a comparison of a 'shopping bag' of items for 1990, with the same items being bought in 1995.

	1990		1995	
Item	Price	Quantity	Price	Quantity
milk	20p	5 pints	33p	7 pints
eggs	85p	3 doz	£1.20	2 doz
coffee	£1.60	1 jar	£2.10	1 jar
bread	36p	4 loaves	55p	6 loaves

Calculate
(a) Paasche's index for prices and
(b) Paasche's index for the number of items, both for 1995, taking 1990 as base.

3. Here are the production quantities (tonnes) and prices (£ per tonne) from a chemical plant.

	1984		1985		1986	
Product	Tonnes	Price (£)	Tonnes	Price (£)	Tonnes	Price
X	6,000	30	6,000	40	6,000	80
Y	4,000	60	2,000	70	5,000	90
Z	2,000	40	3,000	50	4,000	60

Calculate Paasche's price and quantity indices for both 1985 and 1986, taking 1984 as base year each time.

5.4 Comparisons Using Indexes

Suppose we have a manufacturing company which wants to see how it compares with the national situation in its particular sector of manufacturing.

The simplest procedure is to convert its own figures and the national figures into index numbers, with the same (convenient) year being the base year. A quick comparison of like with like will then show the situation.

Example 5.4

The table shows the changes in output value (£m) for a chemical manufacturer Brown Plc over a period of time, and also the values for the national chemical industry (£b) over the same period. How do Brown's figures compare with the national figures?

Year	1995	1996	1997	1998	1999
Brown's (£m)	15.32	15.87	16.03	15.94	16.25
National (£b)	437.2	440.7	445.3	439.6	460.9

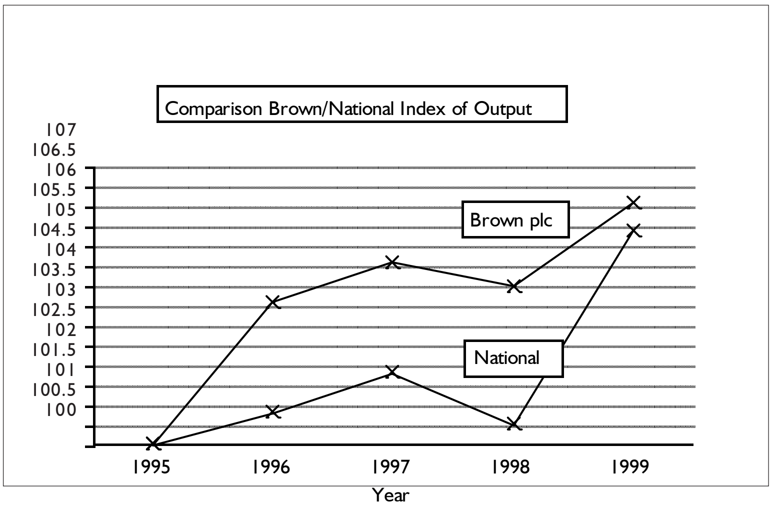
Solution:
Calculate an index for each set of figures, with 1995 = 100 (just as we did on page 74) and we get

Year	1995	1996	1997	1998	1999
Brown's Index	100	103.6	104.6	104.0	106.1
National Index	100	100.8	101.9	100.5	105.4

As you can see, Brown's is doing proportionately better than the national average, with their 1999 output being 6.1% higher than the 1995 output, whereas the national picture shows an increase over the same interval of time as only 5.4%. Brown's weathered the slump of 1998 better than the rest but its recovery wasn't quite so good (perhaps because there wasn't such a slump for them to recover from (?)).

Both sets of figures show a bit of a drop from 1997 to 1998.

It is sometimes better to illustrate figures like these on a graph (but you would not be expected to do this in your assessment).



? 5.4

In each question, calculate index numbers to compare the two sets of figures. Strictly speaking you can make any year the base year, but my answers assume that you've chosen the first year, so perhaps you'd better stick with that.

1. Farmer Bloggs is comparing how his use of pesticides compares with the total use of

Year	1990	1991	1992	1993	1994
Bloggs (tonnes)	8.4	8.7	8.9	8.6	8.2
National (m t)	1.72	1.74	1.75	1.91	2.01

2. Two rival companies making TV equipment are comparing their production levels. Which would you invest in?

Year	1994	1995	1996	1997	1998	1999
A's production	23,600	24,900	25,000	25,800	26,100	26,700
B's production	10,300	10,900	11,400	12,100	12,900	13,800

Now here is a bit for you to read before the last exercise in this section.

5.5 Retail Prices Index

The Retail Prices Index (RPI) is perhaps the best known of all UK official statistics and has general acceptability to both sides of industry and to people of most political persuasions as an indicator of domestic inflation. It is a weighted mean of price relatives index. The current base date is January 1987.

The index represents the average change in prices of millions of consumer purchases, both goods and services, and is based on a 'shopping basket' of 600 specified types of goods and services. Near the middle of the month, as far as possible on a Tuesday, 'price collectors' obtain, on behalf of the Office for National Statistics, about 130,000 prices for some 500 of the indicators in 180 areas throughout the country, always visiting the same shops and buying the same goods so that they can compare like with like. Other prices, such as newspapers and council rents for instance, are more easily obtained centrally. A listing of the various categories of indicators follows.

The components of the RPI are weighted to ensure that the index reflects the importance of the various items in the average shopping basket and the amount we spend in different regions of the country and in different types of shops. These weights, also included in the listing which follows, are obtained from many sources but mainly from the Family Expenditure Survey. The weights are the proportions, expressed as parts of 1,000, of expenditure on each of the 600 items as found by the FES.

The index is kept up to date by the deletion and introduction of various items to reflect changing markets and new products (e.g. black and white TVs dropped in favour of coloured ones). The weights are also changed each year to keep abreast of the general changes in people's spending patterns, e.g. leisure and travel now account for a higher proportion of the family budget than before, and food accounts for a smaller proportion.

Changes in prices are measured by comparing them to their levels the previous January and then weighted together using the latest weights for the current year to produce an overall average price change. The final stage is to link the average price changes with the figures for earlier years. By chain-linking the calculations in this way the index can take account of changes in the make-up of the shopping basket from year to year and provide comparisons with different years.

Structure of the RPI in 1994 (weights out of 1,000 in brackets)

- Food (142)
  bread, cakes, cereals, bacon, beef, lamb, fish, oil, milk, tea, coffee, sweets, fresh fruit & veg, tinned fruit & veg, etc. – a very wide range of foodstuffs
- Catering (45)
  restaurant meals, canteen meals, take-aways, etc.
- Alcoholic Drinks (76)
  beer, wines and spirits
- Tobacco (35)
  cigarettes and other tobacco
- Housing (158)
  rents, council tax, water charges, repairs, d-i-y materials, insurance
- Light & Fuel (45)
  coal, gas, electricity, oil
- Household Goods (76)
  furniture, furnishings, electrical appliances & other equipment, pet care, etc.
- Household Services (47)
  postal & telephone charges, domestic services, fees & subscriptions
- Clothing & Footwear (58)
  men's, women's and children's outerwear, other clothing, footwear
- Personal Items (37)
  personal articles & services, chemist's goods
- Motoring Expenditure (142)
  purchase & maintenance of vehicles, petrol & oil, motor tax & insurance
- Fares (20)
  rail fares, bus & coach fares, other travel costs
- Leisure Goods (48)
  audio-visual equipment, tapes & CD's, toys, photographic, gardening & sports goods, books & newspapers
- Leisure Services (71)
  TV licences & rentals, entertainment & recreation, holidays in UK and abroad

The index was last rebased in 1987 (it had risen to over 300 and was getting a bit remote by this stage, so in 1987 it was scaled back to 100) but the figures for recent years (at January) are:

1995 146.0 1996 150.2 1997 154.4 1998 159.5 1999 163.4

Using the RPI to compare salaries and cost of living

Example 5.5a

Fred's pension is tied to the RPI. In 1996, when the index was 150.2, his pension was £12,490. In 1997 the index went up to 154.4. By how much should his pension go up to keep pace?

Solution:
We do this kind of calculation by proportion.

Index 150.2 →→→→ pension £12,490

Index 154.4 →→→→ pension £12,490 $\times \dfrac{154.4}{150.2}$

= £12,839 (to nearest £)

Fred's pension must rise by £349 just to keep pace with inflation. Although he has more money in his pocket, he can't actually buy more with it because prices have gone up as well.

Example 5.5b

Jenny's hourly rate of pay goes up from £4.82 per hour in 1993 to £5.04 per hour in 1995. Over the same period of time, the RPI has gone up from 137.9 to 146.0 . Is Jenny's pay keeping ahead of inflation, or is she behind?

Solution:
There are several ways to solve this one.

Method 1: as in Example 5.5a, accept the 1993 figures, then see what Jenny's hourly pay should be if she is keeping pace with inflation. The advantage is that she can then see how far ahead or behind she actually is.

Index 137.9 →→→→ rate £4.82

Index 146.0 →→→→ rate £4.82 $\times \dfrac{146.0}{137.9}$

= £5.10 (to nearest 1p)

Since she is only earning £5.04 she has fallen behind and needs another 6 pence per hour to have the same spending power in 1995 as she had in 1993.

Method 2: calculate the ratio 'new÷old' for the wage rate and also for the index. Whichever gives the larger answer is the one which is better (both would be the same if her pay had kept pace with the RPI). The advantage is it's quicker than Method 1.

$$\text{Wage ratio} = \frac{5.04}{4.82} = 1.0456 \ldots \qquad \text{Index ratio} = \frac{146.0}{137.9} = 1.0587 \ldots$$

This shows that Jenny's wages have gone up by 4.56% but the index has gone up by 5.87%, so she has fallen behind. (NB This does NOT mean that she has fallen behind by 1.31%, the calculations are a lot more complicated than that.)

? 5.5

1. In each case calculate what the new wage should be to keep pace with inflation, given the following old wage and the change in RPI over the same interval of time.

   (a) £23,470 a year, old RPI 146.3; new RPI 158.7
   (b) £357.92 per week, old RPI 127.4; new RPI 137.0
   (c) £5.63 per hour, old RPI 157.8; new RPI 169.3

2. Use Method 1 of Example 5.5b to say how far ahead/behind inflation the following employees are. The RPI for 1998 is 159.5 and the RPI for 1999 is 163.4.

   (a) 1998 pay £6.35 per hour, 1999 pay £6.58 per hour
   (b) 1998 pay £312.89 per week, 1999 pay £319.17 per week
   (c) 1998 pay £17,576 per year, 1999 pay £18,312 per year

Answers

? 5.1: Answers

1. 100, 118.7, 122.5, 132.5, 136.8
2. 100, 106.6, 116.3, 102.3, 96.5
3. 64.3, 89.3, 100, 128.6, 150

? 5.2: Answers

(a) Simple aggregative:
 (1) 115.7
 (2) 120.2
 (3) 117.5

(b) Arithmetic mean of price relatives:
 (1) 115.3
 (2) 118.6
 (3) 132.6

? 5.3: Answers

1. (a) 118.5
 (b) 137.5
2. (a) 147.4
 (b) 105.9
3. 1985: price 126.2 quantity 85.5
 1986: price 182.8 quantity 121.9

? 5.4: Answers

1. Bloggs: 100, 103.6, 106.0, 102.4, 97.6
 National 100, 101.2, 101.7, 111.0, 116.9

 By 1994, Bloggs' use is 2.4% down on what he used in 1990, whereas the national figures are showing an increase of 16.9% over the same period of time.

2. (a) 100, 105.5, 105.9, 109.3, 110.6, 113.1
 (b) 100, 105.8, 110.7, 117.5, 125.2, 134.0

 B's actual production is only about half of A's, but is increasing more, so it might be the better bet.

?5.5: Answers

1. (a) £25,459.25
   (b) £384.89
   (c) £6.04
2. (a) £6.51, i.e. 7 pence ahead
   (b) £320.54, i.e. £1.37 behind
   (c) £18,005.76, i.e. £306.24 ahead

6.1 Mean and Standard Deviation

The most common statistic which is used when making comparisons between various sets of numbers is the **mean**.

The mean of a set of numbers is the arithmetic average of those numbers, i.e. the total of the numbers divided by the number of numbers there are.

But knowing just the average of a set of numbers is often not enough. We need to know how the numbers are spread out about the mean.

In fact, the mean can often give a wrong impression about a set of data, and can be dragged out of its expected position by even one number which is much larger, or much smaller, than all the rest.

To give an easy example to illustrate this point, suppose we want to find the mean (or average) of this set of five numbers

15 18 19 19 23

Just looking at them, without any calculations at all, you can probably guess that the mean is round about 19 or so. (I've lined the numbers up in order of size to make the guessing easier.)

The calculation $(15 + 18 + 19 + 19 + 23)/5$ gives us an exact value of 18.8.

Now let's stick an extra large number onto the list – say 150.

The calculation is now $(15 + 18 + 19 + 19 + 23 + 150)/6 = 40.7$

You'll probably agree that 19 is still a better number to represent the whole set of six than 40.7, which is giving a false impression.

A perfect real-life example of this problem with the mean came up at the University of North Carolina recently. A survey of average earnings of its graduates came up with the rather astounding fact that geography graduates were the highest earners, on average. This was very impressive, until closer inspection revealed that one of them had forsaken his geographical career and turned to basketball. As one of its superstars, he currently earns about £40 million (yes, million) each year; hence the inflated average earnings of all geography graduates.

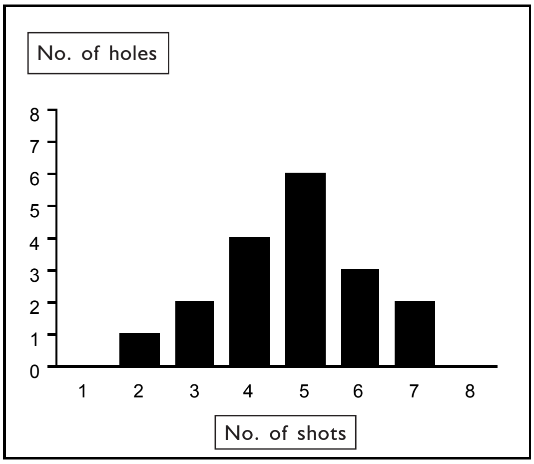
Other statistics have to be calculated which will tell us how the data is spread about. The **standard deviation** is one of these measures of dispersion. It tells you how far away, on average, each member of the data set is from the mean.

Here are two sets of data. Each one has more or less the same mean, or average. But the standard deviation of the first is much less than the standard deviation of the second. This is because the numbers in the first set are clustered together near the mean, but the numbers in the second set are spread about much more.

I have used a player's score card for a round of golf. The table shows that he had no 'holes in one', but one hole required 2 shots, 2 holes required 3 shots each, and so on. The bar chart is shown alongside.

The mean, or average, number of shots per hole is 4.78, which we can get simply by dividing the total number of shots taken (86) by the number of holes in the round (18).

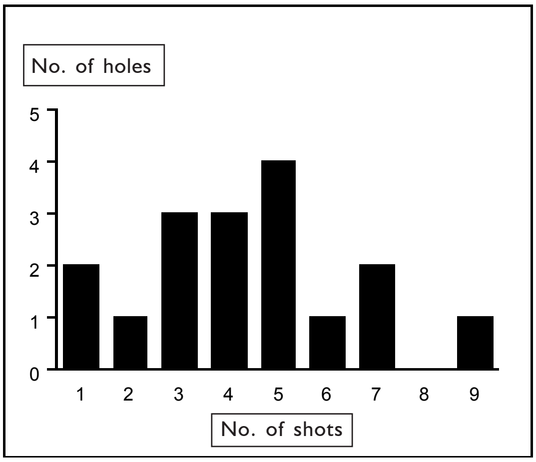
No of shots	No of holes
1	0
2	1
3	2
4	4
5	6
6	3
7	2
8	0



The standard deviation works out at 1.35, but you don't know how to calculate that yet.

This data comes from the score sheet of another golfer. As you see he scored 2 'holes-in-one' but he also took 9 shots at one hole. His performance is very erratic and the scores are far more spread out. The bar chart shows this very clearly.

No of shots	No of holes
1	2
2	1
3	3
4	3
5	4
6	1
7	2
8	0
9	1



The mean of this distribution is 4.61 which is very similar to the mean of the last set. The standard deviation, however, is 2.35, which is close to double the last one.

The mean and standard deviation, used on their own just for one data set, give us a limited summary of what is going on. However, we often have several data sets which we want to compare, and there are various recognised procedures for making comparisons. The means and standard deviations of the data sets play a crucial part in the calculations.

Notation

If the data set is a sample, which it is more often than not, we use the following notation in the Latin alphabet:

Mean: $\bar{x}$ Standard deviation: s

If the data is a population, i.e. it is all the data which exists about a certain situation, then we use the Greek alphabet. This is an internationally recognised convention:

Mean: μ Standard deviation: σ

μ is pronounced 'myoo' and is simply the Greek letter m. σ is pronounced 'sigma' and is the lower case Greek letter s.

(There is also the upper case Greek s, Σ (also pronounced 'sigma'), which is used to mean 'the sum of' and which you will meet very soon. Almost immediately in fact.)

So suppose you are analysing the wages paid to building trades people in a particular company. If you use all of the wages in your calculation then you are dealing with a population and use the Greek letters. On the other hand if some of the data is unavailable and you only use the wages of the workers on one particular site, then you are dealing with a sample and use the Latin letters.

There is actually more to it than that but you don't need to worry about that now. However, I had to bring these two different notations to your attention just in case you were keen and went to a text book to learn more.

Data set a short set of (individual) raw scores

The formulae we use are:

$$\text{Mean: } \frac{\sum x}{n} \qquad \text{Standard deviation: } \sqrt{\frac{\sum (x-\bar{x})^2}{n}}$$

To explain exactly what is going on here: the formula for the mean tells you to add up all the data values (that's the top line) and divide by n which is the letter that represents the total number of data values that there are.

The standard deviation formula is a bit harder to explain. We start off by subtracting the mean ($\bar{x}$) from each member of the data set (x). But this will give us some positive numbers and some negative which will then add up to give us zero. So we square the results of all our subtractions (i.e. we square all these 'deviations from the mean'), add up all these now positive numbers, and find their average by dividing by n.

However, our units are all wrong! Suppose we are analysing wages. The wages are in 'pounds' and we are at the moment calculating an average of 'square pounds' (?) So we take the square root of the answer to bring our units back to the original.

The alternative definition of standard deviation is **the root mean square deviation from the mean** which is precisely what the formula is doing.

Example 6.1

Find the mean and standard deviation of the numbers 3, 7, 2, 7, 4.

Solution:
First find the mean:

$$\bar{x} = \frac{\sum x}{n} = \frac{3+7+2+7+4}{5} = \frac{23}{5} = \mathbf{4.6}$$

Next, make up a table as follows.
The first column is the column of data.
In the second column we subtract the mean from each number in the first column.
In column 3 we square each number in column 2, then add up all these squares.

x	$x-\bar{x}$ $= x-4.6$	$(x-\bar{x})^2$
3	–1.6	2.56
7	2.4	5.76
2	–2.6	6.76
7	2.4	5.76
4	–0.6	0.36
TOTAL	0	21.20

$$s = \sqrt{\frac{(x-\bar{x})^2}{n}}$$

$$= \sqrt{\frac{21.20}{5}} = \sqrt{4.24} = \mathbf{2.06} \text{ (to 2 d. p.)}$$

i.e. on average, each number is 2.06 away from the mean of 4.6.
(The 0 total in the table just helps to check that the column entries are OK.)

? 6.1

Calculate the mean and standard deviation of these sets of numbers. Before you start, make a guess as to what you think the answers will be.

1. 5 6 8 8 9 12
2. 7 8 8 8 10 10 12

In the next question, round off your mean to 1 decimal place before you find the standard deviation.

3. 3.5 4.2 4.2 4.3 4.9 5.0 5.2 5.4

6.2 Data in the Form of a Simple Frequency Table

We have to amend our formulae slightly

$$\bar{x} \text{ or } \mu = \frac{\sum fx}{\sum f} \qquad s \text{ or } \sigma = \sqrt{\frac{\sum fx^2}{\sum f} - \left(\frac{\sum fx}{\sum f}\right)^2}$$

Example 6.2

Here we have the number of strokes required by golfers to get round a particular course one day:

No. of Strokes (x)	67	68	69	70	71	72	73	74	75
No. of Golfers (f)	15	23	38	27	25	19	16	10	4

Find the mean and standard deviation.

Solution:
How do we apply the formulae to this table?

We start by writing the data in a column, not a row, and adding a column for fx and another column for fx^2.

Note that fx means $f \times x$ and that fx^2 means either $fx \times x$ or $f \times x^2$ but emphatically NOT $fx \times fx$.

Thus the formula for the mean states '(total number of strokes) ÷ (total number of golfers)' giving us the average number of strokes per golfer.

You need some fairly complicated maths to see that the formula for the standard deviation calculates the same thing as the first formula did on page 92, but please take my word for it that it does.

x	f	fx	fx^2
67	15	1,005	67,335
68	23	1,564	106,352
69	38	2,622	180,918
70	27	1,890	132,300
71	25	1,775	126,025
72	19	1,368	98,496
73	16	1,168	85,264
74	10	740	54,760
75	4	300	22,500
TOTALS	177	12,432	873,950

Thus the first line reads:

15 golfers each scored 67.
This makes a total of $15 \times 67 = 1{,}005$ strokes.
The 67,335 we get by multiplying 1,005 by 67, alternatively, $15 \times 67 \times 67$ or 15×67^2.

Other lines you get in a similar way.
Then find the total of each column.

Using the memory keys on your calculator speeds up the calculations considerably. But since memories can be operated in many different ways, ask your tutor about this. If all else fails, read the instructions.

Before you do the calculations, look at the data and see if you have some sort of idea as to the answers you will get. You see there is a quick build-up of numbers to 69, then a long tail to 75. You could make a rough guess that the mean will be about 70.

The standard deviation is also easy to estimate. Take the range of your data (= the largest minus the smallest member of the data set), in this case 75 – 67 = 8. Now divide this 8 by 4 (answer 2) and divide the 8 by 3 (answer 2.7). In many situations, the standard deviation should end up between these two answers, i.e. between 2 and 2.7 .

Now let's do the actual calculations:

$$\bar{x} = \frac{\sum fx}{\sum f} = \frac{12{,}432}{177} = 70.24 \qquad s = \sqrt{\frac{\sum fx^2}{\sum f} - \left(\frac{\sum fx}{\sum f}\right)^2} = \sqrt{\frac{873{,}950}{177} - \left(\frac{12{,}432}{177}\right)^2} = 2.07$$

As you see, both tie in with our initial estimates. Making estimates like this won't help if you make a small error in the calculations, but they do tell you if you are miles out.

Some people find calculating the standard deviation on the calculator a difficult thing to do.

If you have a modern scientific calculator, with DAL or VPAM on it, key the numbers as follows:

[√] [(] 873950 [÷] 177 [−] [(] 12432 [÷] 177 [)] [x^2] [)] [=]

and you should get the answer.

If you have an older scientific calculator, follow the same sequence but put the square root sign at the end instead:

[(] 873950 [÷] 177 [−] [(] 12432 [÷] 177 [)] [x^2] [)] [√]

If you have a simple, basic £2.99 job, do the first division (i.e. the 873950/177) and write the answer down. Then do the second one, multiply it by itself, and write the answer down.

Next, subtract the second answer from the first answer, then square root the final answer: laborious, but it will work.

? 6.2

1. Below is a table which gives the breaking stress of each of 120 test cubes of concrete, the units of measurement being in N/mm^2. The figures have been rounded off to the nearest whole unit. Calculate the mean breaking stress and the standard deviation.

Breaking Stress	23	24	25	26	27	28	29	30	31
No. of Cubes	2	5	6	9	17	28	32	14	7

2. In absorption tests on 200 bricks the following figures were obtained. Calculate the mean absorption rate and the standard deviation.

% absorption	6	7	8	9	10	11	12	13	14	15
frequency	1	5	13	31	51	47	33	14	4	1

3. The number of hours worked each week by each of 100 workers in a construction company was recorded as follows to the nearest hour. Calculate the mean number of hours worked and the standard deviation.

Hours per man	40	41	42	43	44	45	46	47	48
Number of men	3	4	8	12	17	21	19	11	5

6.3 Data in the Form of a Grouped Frequency Table

Example 6.3a

Here we see the wages of a set of employees, grouped in intervals of £50 (apart from the last one):

Wages (£)	100-150	150-200	200-250	250-300	300-350	350-400	400-500
Frequency (*f*)	17	25	34	47	26	15	3

Find the mean and standard deviation.

Solution:
The first interval thus encompasses all wages between £100.00 and £149.99. A wage of £150.00 is included in the second interval.

Consider the first interval. The 17 employees could all be earning as much as £149.99 each or as little as £100.00 each, we have no way of knowing. So, in order to make any calculations at all, we have to make a fairly big assumption – that they all earn an amount slap-bang in the middle of the interval, i.e. $(100+150)\div2 = £125$.

The table now looks like this, the second column (our *x* column) being the **mid-values** of each interval. That the last interval is different makes no odds, our last mid-value just looks a bit different from the rest as well.

Wages	Mid value *x*	*f*	*fx*	fx^2
100-150	125	17	2,125.0	265,625.00
150-200	175	25	4,375.0	765,625.00
200-250	225	34	7,650.0	1,721,250.00
250-300	275	47	12,925.0	3,554,375.00
300-350	325	26	8,450.0	2,746,250.00
350-400	375	15	5,625.0	2,109,375.00
400-500	450	3	1,350.0	60,7500.00
TOTALS		167	42,500.0	11,770,000.00

The calculations now follow:

$$\bar{x} = \frac{\sum fx}{\sum f} = \frac{42{,}500}{167} = £254.49$$

$$s = \sqrt{\frac{\sum fx^2}{\sum f} - \left(\frac{\sum fx}{\sum f}\right)^2} = \sqrt{\frac{11{,}770{,}000}{167} - \left(\frac{42{,}500}{167}\right)^2} = £75.59$$

The mean wage of this group of employees is £254.49 per week, and the wages are spread about so that, on average, the wages are £75.59 away from the mean.

The real mean and standard deviation will be a bit different, of course, but these figures will be close enough to be used in any further calculations or discussions.

Example 6.3b

Here we have the same workforce, but the intervals have been defined slightly differently.

Wages (£)	100-149	150-199	200-249	250-299	300-349	350-399	400-499
Frequency (*f*)	17	25	34	47	26	15	3

Find the mean and standard deviation.

Solution:

The first interval starts £1 short of where the second one starts. In other words, the 17 people in the first interval earn between £99.50 and £149.50, the 25 people in the second interval earn between £149.50 and £199.50 and so on.

Thus 149.5 is the boundary between the first interval and the second interval. Similarly, 199.50 is the boundary between the second interval and the third. And so on.

Using the previous convention, someone earning exactly the boundary of £149.50 will end up in the second interval and not the first.

The mid-values are calculated as before – the first one is (100 + 149)/2 = 124.5 and so on.

Wages	Mid value *x*	*f*	*fx*	fx^2
100-149	124.5	17	2,116.5	263,504.25
150-199	174.5	25	4,362.5	761,256.25
200-249	224.5	34	7,633.0	1,713,608.50
250-299	274.5	47	12,901.5	3,541,461.75
300-349	324.5	26	8,437.0	2,737,806.50
350-399	374.5	15	5,617.5	2,103,753.75
400-499	449.5	3	1,348.5	606,150.75
TOTALS		167	42,416.5	11,727,541.75

$$\bar{x} = \frac{\sum fx}{\sum f} = \frac{42{,}416.5}{167} = £253.99$$

$$s = \sqrt{\frac{\sum fx^2}{\sum f} - \left(\frac{\sum fx}{\sum f}\right)^2} = \sqrt{\frac{11{,}727{,}514.75}{167} - \left(\frac{42{,}416.5}{167}\right)^2} = £75.59$$

Compare these two answers to those in Example 6.3a.

What's the connection? What's the explanation?

Answers at the end.

? 6.3

1. The following table gives the moisture content in 100 test samples of a particular cement mix after a certain time interval. Calculate the mean moisture content and the standard deviation.

% moisture content	20-25	25-30	30-35	35-40	40-45	45-50
Number of samples	5	17	52	18	7	1

2. The table represents the weights of a sample of 100 female students at a college. Find the mean weight and the standard deviation.

Weight (kg)	60-62	63-65	66-68	69-71	72-74
Frequency	5	18	42	27	8

3. The compressive strengths of a sample of 30 bricks was measured and listed in the following table. Find the mean and standard deviation.

Strength	20-21	21-22	22-23	23-24	24-25	25-26	26-27	27-28
No. of bricks	2	3	4	5	7	4	3	2

4. The number of cars crossing a toll bridge every day was noted over a period of a month. Find the mean and standard deviation. The mid-values will all be 'something.5', and I know that you can't get 0.5 of a car, but follow the procedure anyway. Similarly, the mean and standard deviations will both have decimal places in them. That's OK.

No. of Cars	500-599	600-699	700-799	800-899	900-999	1,000-1,099
No. of Days	1	4	3	14	7	2

Answers

? 6.1: Answers

Remember that, in questions 1, 2 and 3, you have to calculate $\bar{x}$ before you can start filling out the second column of the table.

1.

x	$x - \bar{x}$ $= x - 8$	$(x - \bar{x})^2$
5	−3	9
6	−2	4
8	0	0
8	0	0
9	1	1
12	4	16
$\sum x = 48$	0	$\sum (x - \bar{x})^2 = 30$

$$\bar{x} = \frac{48}{6} = 8$$

$$s = \sqrt{\frac{30}{6}} = \sqrt{5} = 2.24$$

2.

x	$x - \bar{x}$ $= x - 9$	$(\bar{x} - x)^2$
7	−2	4
8	−1	1
8	−1	1
8	−1	1
10	1	1
10	1	1
12	3	9
$\sum x = 63$	0	$\sum (x - \bar{x})^2 = 18$

$$\bar{x} = \frac{63}{7} = 9$$

$$s = \sqrt{\frac{18}{7}} = 1.60$$

Note that the standard deviation of question 2 is smaller than that of question 1. This is as expected. The data of question 2 is spread out less than the data of question 1.

3.

x	$x - \bar{x}$ $= x - 8$	$(x - \bar{x})^2$
3.5	−1.1	1.21
4.2	−0.4	0.16
4.2	−0.4	0.16
4.3	−0.3	0.09
4.9	0.3	0.09
5.0	0.4	0.16
5.2	0.6	0.36
5.4	0.8	0.64
$\sum x = 36.7$	−0.1	$\sum (x - \bar{x})^2 = 2.87$

$$\bar{x} = \frac{36.7}{8} = 4.6$$

$$s = \sqrt{\frac{2.87}{8}} = 0.60$$

The data in question 3 is even more squashed up than in question 1 or 2, so the standard deviation is even smaller.

The −0.1 total of column 2 should be 0, but the rounding off was unfortunate. Rounding to 2 decimal places would have been better, perhaps, but it would also have taken longer.

Actually, the final answer for s is remarkably close to the accurate one.

? 6.2: Answers

1.

x	f	fx	fx^2
23	2	46	1,058
24	5	120	2,880
25	6	150	3,750
26	9	234	6,084
27	17	459	12,393
28	28	784	21,952
29	32	928	26,912
30	14	420	12,600
31	7	217	6,727
Totals	$\sum f = 120$	$\sum fx = 3,358$	$\sum fx^2 = 94,356$

$$\bar{x} = \frac{3,358}{120} = 27.98 \text{ N/mm}^2$$

$$s = \sqrt{\frac{94,356}{120} - \left(\frac{3,358}{120}\right)^2}$$

$$= 1.80 \text{ N/mm}^2$$

2. $\sum f = 200$ $\sum fx = 2,100$ $\sum fx^2 = 22,538$

   $\bar{x} = 10.5\%$ $s = 1.56\%$

3. $\sum f = 100$ $\sum fx = 4,460$ $\sum fx^2 = 199,284$

   $\bar{x} = 44.6$ hrs $s = 1.92$ hrs

Pages 97–8

If you look at the mid-values of Example 6.3b they are all 0.5 less than the mid-values of Example 6.3a.

So, if you imagine the histogram of 6.3a, all you have to do is shift the whole lot, all the columns in one heap, by 0.5 to the left, and you'll get the histogram of 6.3b.

It's easy to see, then, that the mean has been shifted by 0.5 to the left as well. That's why the mean in 6.3b is exactly 0.5 less then the mean in 6.3a, even though the numbers involved in its calculation are different.

As regards the standard deviation, the spread of the data (the shape of the histogram, if you like), has not changed as you moved it left. That's why the standard deviation is exactly the same in 6.3b as it is in 6.3a.

? 6.3: Answers

1. Using as mid-values 22.5, 27.5, ..., 47.5 we get

   $\sum f = 100$ $\sum fx = 3{,}290$ $\sum fx^2 = 110{,}525$

   $\bar{x} = 32.9\%$ $s = 4.78\%$

2. Using as mid-values 61, 64, 67, 70, 73 we get

   $\sum f = 100$ $\sum fx = 6{,}745$ $\sum fx^2 = 455{,}803$

   $\bar{x} = 67.45$ kg $s = 2.92$ kg

3. Using as mid-values 20.5, 21.5, ..., 27.5 we get

   $\sum f = 30$ $\sum fx = 721$ $\sum fx^2 = 17{,}435.5$

   $\bar{x} = 24.03$ units $s = 1.89$ units

4. Using as mid-values 549.5, 649.5, ..., 1,049.5 we get

   $\sum f = 31$ $\sum fx = 26{,}034.5$ $\sum fx^2 = 22{,}291{,}457.75$

   $\bar{x} = 839.8$ cars $s = 117.4$ cars

If you think about the burning question again from page 98, you might come up with a slightly quicker way to work out the answer to question 4.

SECTION 7

You may have noticed that, when using a scientific calculator, your answer sometimes looks like this:

1.23456 –05 or 1.23456^{-05}

The calculator sometimes defaults automatically into what is called **scientific notation** or **standard form** for answers to calculations which are either very high or very low.

A number appears in scientific notation when it is written in the form $a \times 10^n$
where the number *a* must be between 1 and 10.

7.1 Large Numbers – Powers of 10

You know that the power of a number tells you how many times that number is multiplying itself. For example:

$2^3 = 2 \times 2 \times 2 = 8$
$3^4 = 3 \times 3 \times 3 \times 3 = 81$

Here is a summary of powers of 10:

10^1	10^2	10^3	10^4	10^5	10^6
10	100	1,000	10,000	100,000	1,000,000

You will notice that the power of 10 is the same as the number of zeros. For instance, 10^3 equals 1,000 which has 3 zeros. And multiplying a number by 1,000 moves the decimal point 3 places to the right, i.e. the same as the number of zeros.

Thus we can change any number which is written in scientific notation into its equivalent written conventionally:

Example 7.1a
Write the following numbers conventionally: 5×10^3 2.4×10^6 5.79×10^4

Solution:
$5 \times 10^3 = 5,000$
$2.4 \times 10^6 = 2,400,000$
$5.79 \times 10^4 = 57,900$

Explanation: $5 \times 1,000 = 5,000$ is easy for the first one.

As for the second, you have to move the point 6 places to the right because of the power 6, so you need to 'invent' 5 zeros for it to jump over. Alternatively, the whole number 2 is multiplied to give a million, and the 4 has to follow into the next place (the hundred thousand column), and we need to fill the other columns with zeros as place holders.

If after a calculation your calculator display shows something like 1.2345678 05 you now know that this means 1.2345678×10^5 and you have to move the point 5 places right to get 123,456.78.

Whether you want all these digits or not is another matter. It may make sense to round off the answer to 123,000 or 123,500 but this will depend on the context of the question.

Going backwards is almost as easy.

Example 7.1b

Express 5,320 in scientific notation.

Solution:
You must put the point between the 5 and the 3 because the rule states that in the format $a \times 10^n$ the value of a must be between 1 and 10.

This means that you have to jump the point 3 places to the right to get it to the end of the number (since 5,320 is a whole number, the point sits invisibly at the end, after the 0.)

So we get the final answer 5.32×10^3 (you can write 5.320×10^3 if you want, the zero at the end won't change the number).

Example 7.1c

Express 154,738,296.17 in scientific notation.

Solution:
Since this number is well over a million, it is doubtful if the .17 at the end matters very much. Equally, the 296 at the end is probably not very significant either. So we can start by rounding the number off to something sensible: 154,738,000.

In the format of scientific notation, the decimal point must be placed between the 1 and the 5, and we then have to jump the point 8 places to the right to get it to its rightful position.

So the answer is 1.54738×10^8 though further rounding might make 1.55×10^8 equally acceptable.

Example 7.1d

Here are a few more examples. Check to see why the conversions are correct. Some of the 'longer' numbers have been rounded off to 2 decimal places.

$$64{,}000 = 6.4 \times 10^4$$
$$128{,}000 = 1.28 \times 10^5$$
$$4{,}000{,}000{,}000 = 4.0 \times 10^9$$
$$547{,}902 = 5.48 \times 10^5$$
$$692.714 = 6.93 \times 10^2$$
$$1{,}638.19834 = 1.64 \times 10^3$$
$$27{,}813.087 = 2.78 \times 10^4$$

There are a few calculators which will convert from one format to another, but with most, at the time of writing, you have to convert yourself.

? 7.1

These numbers are written in scientific notation. Write them conventionally.

1. 2.36×10^5 2. 5.90×10^3 3. 6.05×10^8

Each of the next set of questions shows a calculator display. Write each number in its conventional format.

4. 4.183 05 5. 7.187 08 6. 4.190298374 04

Each of the next set of questions shows a number written conventionally. Rewrite each number in scientific notation.

7. 2,400 8. 2,153,600 9. 819.7

Rewrite each of the following numbers in scientific notation, but round each answer to 2 decimal places.

10. 53,781.23 11. 604.19283 12. 213,542.61

That takes care of large numbers.

Now you must learn how to write small numbers in scientific notation. This is particularly useful for anyone studying sciences, and also business and accountancy, since calculators can often default into scientific notation without any effort on your part!

7.2 Small numbers – powers of 10

By small we mean numbers between 0 and 1, i.e. numbers starting with 'zero point' something.

We can extend the table of powers of 10 into small numbers which are fractions of 1:

$\frac{1}{10,000}$	$\frac{1}{1,000}$	$\frac{1}{100}$	$\frac{1}{10}$	1	10	100	1,000
10^{-4}	10^{-3}	10^{-2}	10^{-1}	10^{0}	10^{1}	10^{2}	10^{3}

As you go from right to left along the top line of the table you see that you are dividing by 10.

As you go from right to left along the bottom line of the table you see that the powers of 10 are decreasing by 1 each time.

You eventually reach $10^0 = 1$ and then negative powers:

$10^{-1} = 0.1$
$10^{-2} = 0.01$
$10^{-3} = 0.001$ and so on.

Notice that the power is the same as the number of zeros at the start of the number with the decimal point after the first zero.

The negative power does NOT mean that the number is negative. It means that the number is SMALL, and the more negative the power, the smaller the number.

Example 7.2a

Express the number 3.12×10^{-4} in conventional form.

Solution:
We can rewrite first of all as 3.12×0.0001 and the answer to this is 0.000312. Again, note that the total number of zeros at the start of the number is 4, with the point after the first one.

This gives us a quick method of rewriting without having to think too much about it !

Example 7.2b

Express the number 8.45×10^{-6} in conventional format.

Solution:
The power is -6 so start by writing six zeros followed by 845: 000000845.
Then put a decimal point after the first zero: 0.00000845 is the answer. Rewriting the other way simply reverses the procedure.

Example 7.2c

Rewrite the number 0.00614 in scientific notation.

Solution:
The number is less than 1 so the power of 10 will be negative. It starts off with a total of 3 zeros, so the power will be -3.
The number *a* in the scientific notation format must be between 1 and 10 so we put the point between the 6 and the 1.

Final answer: 6.14×10^{-3}

In the next example we have the problem of rounding off an answer which has too many digits, i.e. is too detailed.

Example 7.2d

Rewrite the number 0.0000000091762354 in scientific notation, and round off the final answer to 2 decimal places.

Solution:
We have a total of 7 zeros, so the power of 10 will be -7. To conform with the rules we have to place the point between the 9 and the 1.
The number 9.1762354 will round off to 9.18 (this is more mathematically called 'rounding off to 3 significant figures' rather than '2 decimal places')

So the final answer is 9.18×10^{-7}.

Example 7.2e

Here are several assorted conversions, all rounded off to 3 significant figures:

$$2.38 \times 10^{-4} = 0.000238$$
$$5.9732 \times 10^{-2} = 0.0597$$
$$6.76543 \times 10^{-3} = 0.00677$$
$$2.01978 \times 10^{-4} = 0.000202$$
$$8.99681 \times 10^{-5} = 0.0000900$$

In the last one, 0.0000900 is correct to 3 significant figures (the 9 and the two zeros which follow it), but 0.009 would be correct to only 1 significant figure (the 9). Strictly speaking they are not the same. The zeros following the 9 show that 0.0000900 is a much more accurate number than just 0.00009.

But don't worry about it !

Now for some practice.

? 7.2

Rewrite this first set in conventional form.

1.	2.37×10^{-4}	2.	5.28×10^{-6}	3.	6.09×10^{-3}

Here are some 'calculator displays'. Convert each one into conventional form, and round the answer to 3 significant figures.

4.	4.98765 –03	5.	3.06134 –05	6.	5.13254 –02

Here are some numbers written conventionally. Rewrite them in scientific notation, and round off answers to 3 significant figures.

7.	0.00725	8.	0.08531	9.	0.000765432

In this final set there are some large numbers mixed in as well. Rewrite each number in its alternative format.

10.	43,400	11.	6.15×10^{-2}	12.	0.0000765
13.	8.31×10^{-4}	14.	451,000	15.	8.2×10^{6}
16.	7.9×10^{-6}	17.	4,270,000	18.	0.000000832

7.3 Manipulating using Scientific Notation

All scientific calculators have a 'scientific' mode, in which all calculations will be performed using scientific notation. What's more, you can usually specify the number of significant figures you want to use.

It is often accessed by pressing a MODE key followed by some sort of number code (e.g. MODE 6) followed by the number of figures you want to keep in your answers. Other calculators have a key marked ESF or EFS or something similar.

If in doubt, consult the manual.

You can enter numbers in their normal, conventional format, perform some operations, and the answer will pop out in scientific notation.

Or you can enter the numbers in scientific notation to start with.

To do this you use the key marked EXP (or EE on some older machines).

When you press the EXP key you are actually typing in 'times 10 to the power of'.

Thus to key in 5.56×10^{-3} you actually key 5.56 (EXP) –3. You DON'T type the multiplication sign. To type the 'minus' sign you have to press a key which looks either like this (–) or like this (+/–)

Example 7.3a

Calculate the value of AB when A = 5.73×10^{-2} and B = 2.18×10^{-3}.

Solution:
Watch carefully how you key the problem in:

5.73 (EXP) (–) 2 (×) 2.18 (EXP) (–) 3 (=)

Notice that we are multiplying the two numbers; we only have to key in ONE multiplication sign. The EXP key already contains the times sign within each number.
The answer, incidentally, is 1.249×10^{-4}.

Example 7.3b

Evaluate $\dfrac{p}{q + r}$ where $p = 2.6 \times 10^3$, $q = 4.2 \times 10^{-4}$ and $r = 5.7 \times 10^{-5}$

Solution:
Because you have a calculation within the fraction you must put brackets round the addition on the bottom line, otherwise the calculator will divide p by q, then add r to the answer, which is not what you want.

2.6 (EXP) 3 (÷) (() 4.2 (EXP) (–) 4 (+) 5.7 (EXP) (–) 5 ()) (=)

The answer is 5.451×10^6.

Example 7.3c

A rocket is flying through space at a speed of 3.27×10^4 km per hour. How many seconds does it take to pass two asteroids which are 'stationary' relative to the rocket and which are 534 km apart?

Solution:

The speed is given in kilometres per hour, but we want the time in seconds, so we have to divide the speed by 3,600 to convert it to km per second. This is because there are $60 \times 60 = 3{,}600$ seconds in an hour, and it will travel less in one second than it will in one hour, hence the division.

$$\text{Time} = \frac{\text{Distance}}{\text{Speed}}$$

$$= \frac{534 \text{ km}}{(3.27 \times 10^4 \div 3600) \text{ km/s}}$$

The calculator sequence is:

534 [÷] [(] 3.27 [EXP] 4 [÷] 3600 [)] [=] and the calculator display shows 5.879 01.

This means 5.879×10^1 which translates into 58.79 seconds.

? 7.3

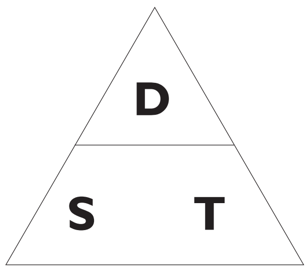
Put your calculator into scientific mode with 4 significant figures, then carry out the following calculations.

NB The questions are typed with brackets round each number to keep the bits of it together. You only need to put brackets in where the calculator would make a mistake if they weren't there, i.e. the bottom line of questions 7 and 8.

1.	$(2.5 \times 10^3) \times (3.0 \times 10^{-4})$	2.	$(1.4 \times 10^{-2}) \times (3.6 \times 10^4)$
3.	$(7.827 \times 10^3) \div (4.014 \times 10^8)$	4.	$(6.7 \times 10^{-2}) \div (4.38 \times 10^{-5})$
5.	$\dfrac{(2.3 \times 10^2) \times (5.6 \times 10^{-5})}{7.2 \times 10^{-4}}$	6.	$\dfrac{(5.9 \times 10^{-5}) \times (6.4 \times 10^{-7})}{3.1 \times 10^{-2}}$
7.	$\dfrac{4.3 \times 10^2}{(2.3 \times 10^5) \times (6.4 \times 10^7)}$	8.	$\dfrac{(4.2 \times 10^8) \times (6.9 \times 10^5)}{(1.2 \times 10^7) \times (8.3 \times 10^6)}$

The next few questions involve speed, distance and time.

You may remember this triangle from school. You cover up the quantity you want, and the diagram shows you what to do with the others.



'**D**ustbins **S**mell **T**errible' may help you remember this!

If you want the Distance, cover up D.
The diagram shows $S \times T$.

If you want the Speed, cover up S.

The diagram shows $\dfrac{D}{T}$.

If you want the Time, cover up T.

The diagram shows $\frac{D}{S}$.

9. (a) How many seconds are there in a year (365 days)?
   (b) A light year is the distance light would travel in one year. If the speed of light is 3×10^8 metres per second, how many kilometres is one light year?
   (c) A star is at a distance of 4.8 light years from us. How far away is that?
   (d) The fastest rocket at present can travel at about 5×10^4 km per hour. How long would it take to travel to the star in (c) above?

10. An asteroid is travelling at 2.3×10^3 km per hour. How far does it travel in one year?

11. How long does it take for a rocket travelling at 5.4×10^3 km per hour to cover a distance of 6.2×10^{15} km?

12. (a) Convert a speed of 2.4×10^3 metres per second into kilometres per hour.
    (b) Convert a speed of 5.0×10^5 kilometres per hour into metres per second.

13. A comet covers a distance of 50 million million kilometres in one year. What is its speed in metres per second?

14. An atom has a diameter of 3×10^{-14} metres. How many atoms arranged side by side would cover the distance to the moon of approximately 3.5×10^5 kilometres?

15. Find the time taken in days for a particle travelling at 4.8×10^3 metres per second to cover a distance of 12×10^8 kilometres.

16. There are an estimated 2×10^{11} stars in an average galaxy and 1.5×10^{11} galaxies in the universe. Find the estimated number of planets (this includes asteroids and miscellaneous floating rocks) in the universe if each star has an average of 1.1×10^2 planets.

7.4 Evaluating Formulae

Example 7.4a: Showing use of the x^2 key

This key squares a number, i.e. multiplies it by itself.

An obvious application with which you should be familiar is the area of a circle: $A = \pi r^2$.

If the radius is, say, 5.8 cm then the area will be $\pi \times 5.8^2$ cm^2 (squared centimetres).

The sequence for this is (π) (×) 5.8 (x^2) (=) [Answer 105.68 cm^2]

Another obvious example is the Pythagoras formula, which may involve a calculation such as

$$x = \sqrt{4.3^2 + 5.9^2}$$

for which the calculator sequence is (√)(() 4.3 (x^2)(+) 5.9 (x^2)())(=)
[Answer 7.3]

Example 7.4b: Showing use of the (x^y) or (y^x) or (^) key
(Some calculators show it one way, some another, all are used in the same manner.)

This key shows powers and is used for powers other than 2. An obvious example is the formula for the volume of a sphere:

$$V = \frac{4}{3}\pi r^3$$

Suppose the radius of the sphere is 5.8 cm, the evaluation will be

4 (x) (π) (x) 5.8 (^) 3 (÷) 3 (=) [Answer 817.28 cm^3]

NB The power key is higher in the order of priorities than all the others.

For instance if you want to evaluate $2^3 + 4$ you key 2 (^) 3 (+) 4 (=) [Answer 12]. The '2 to the power 3' is evaluated before the addition of the 4.

But if you want 2^{3+4} you have to put the whole power in brackets
2 (^)(() 3 (+) 4 ()) (=) for which the answer is 128.

As you see, this section really involves slightly higher-order calculator skills, with a wee bit about common sense units thrown in. If the context of the formula is totally alien to you, it doesn't actually matter at all; you can still carry out the evaluation.

The most important thing about evaluating a formula is to first write out the formula with numbers replacing the variables, and only then reach for the calculator. This is to avoid mistakes.

Example 7.4c

This example is from the world of optics. If *f* is the focal length of a lens, *u* is the object distance and *v* is the image distance (all in identical units, i.e. all in mm or all in cm), these three quantities are related by the formula

$$\frac{1}{f} = \frac{1}{v} + \frac{1}{u}$$

Find *f* when $v = 5$ cm and $u = 10$ cm.

Solution:

When we substitute the values we get $\frac{1}{f} = \frac{1}{5} + \frac{1}{10}$, the calculator sequence for the right hand side of this being either 1 (÷) 5 (+) 1 (÷) 10 (=) or 5 (x^{-1}) (+) 10 (x^{-1}) (=) (answer 0.3 either way).

NOTE the (x^{-1}) key which gives you 'one over a number', also known as the **reciprocal**.

But the answer so far is not for *f* but for 'one over *f*' so we have to press the (x^{-1}) key again to get the final answer 3.333 (centimetres to be consistent with the other units.)

Example 7.4d

Evaluate the formula $T = 2\pi\sqrt{\frac{l}{g}}$ where g = 9.81 m s^{-2} (the acceleration due to gravity), l = 25 cm (length of a pendulum) and T is the time in seconds for the pendulum to complete one swing.

Solution:

The force of gravity is given in metres per second per second but the length of the pendulum is given in centimetres. Common sense tells us to rewrite *l* as 0.25 metres before substitution in the formula.

The formula $T = 2\pi\sqrt{\frac{l}{g}}$ becomes $T = 2\pi\sqrt{\frac{0.25}{9.81}}$ for which the calculator sequence is

2 (×) (π) (×) (√) (() 0.25 (÷) 9.81 ()) (=) and the answer is 1.003 (seconds).

The brackets within the square root are necessary so that the root knows there is a calculation within it. Without the brackets only the 0.25 would have been rooted, and the whole answer divided by 9.81.

If you have a slightly older scientific calculator, you need to have a number on the display before you can tell the calculator to do something. Hence the root sign has to come after the division and not before! So the sequence is

(() 0.25 (÷) 9.81 ()) (√) (×) 2 (×) (π) (=)

Note the use of the (π) key. There is not much point typing in 3.14 if there is a key for π which gives it more accurately anyway.

Example 7.4e: an example from the study of electricity

A 30-μF capacitor is connected across a 400 V, 50 Hz supply. Calculate the current using the formula.

Capacitive Reactance, $X = \dfrac{1}{2\pi f C}$ Ohms and Current, $I = \dfrac{V}{X}$ Amps

where V is the voltage, f is the supply frequency and C is the capacitance.

Solution:
This is a highly specialised formula and, unless you are an expert in electricity, you are not expected to know what's what. So don't panic!

You should, however, know that just as the prefix 'm' or 'milli' stands for 'one thousandth part of', i.e. a milliunit = unit $\times 10^{-3}$, so the prefix μ, read 'micro', stands for 'one millionth part of', i.e. a microunit = unit $\times 10^{-6}$.

Substituting into the formulae, then, we get

$$X = \frac{1}{2 \times \pi \times 50 \times 30 \times 10^{-6}} = 106.103\ ...\Omega \text{ (Ohms) followed by}$$

$$I = \frac{400}{106.103} = 3.77 \text{ A}$$

The best calculator sequence for this on a modern calculator is

2 (×) (π) (×) 50 (×) 30 (EXP) (–) 6 (=) (x^{-1}) (=) followed by 400 (÷) (ANS) (=)

The (ANS) key automatically keys in the answer to your last calculation (the 106.103), thus saving you the effort of retyping the number (and possibly getting it wrong).

? 7.4

Evaluate the following formulae:

1. $V = E - IR$ when $I = 14.5$, $R = 0.36$, $E = 240$

2. $L = \dfrac{xy^2}{z}$ when $x = 4.7$, $y = 5.4$, $z = 25.2$

3. $P = T\sqrt{\dfrac{m}{n+p}}$ when $m = 15.8$, $n = 3.4$, $p = 2.7$, $T = 6.4$

4. $K = \dfrac{Wv^2}{2g}$ when $v = 4.7$, $W = 12.6$, $g = 9.81$

5. $X = \dfrac{1}{2\pi fC}$ when $f = 50$, $C = 2.3 \times 10^{-4}$

6. $V = \dfrac{2R}{R+r}$ when $R = 7.3$, $r = 1.4$

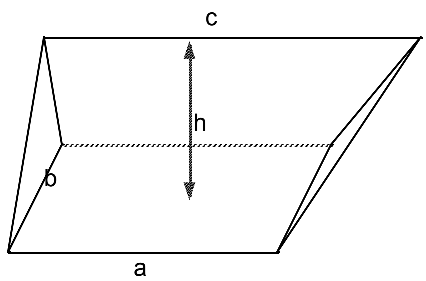
7. $A = \dfrac{1}{2}m(v^2 - u^2)$ when $v = 7.3$, $u = 2.4$, $m = 5.6$

8. $f = \dfrac{1}{2\pi\sqrt{Lc}}$ when $L = 0.03$, $c = 5.6 \times 10^{-6}$

9. $R = \dfrac{R_1R_2}{R_1 + R_2}$ when $R_1 = 3.7$, $R_2 = 8.5$

10. $T = \dfrac{1}{2\pi}\sqrt{\dfrac{l}{g}}$ when $l = 25$, $g = 9.81$

11. The formula for the volume of this wedge is



$V = \dfrac{(2a + c)bh}{6}$

Find the volume in cm^3 if

$a = 7.2$ cm, $b = 6.4$ cm
$c = 8.3$ cm, $h = 26$ mm

(Watch – the units must be consistent.)

12. The formula for the volume of a sphere is $V = \dfrac{4}{3}\pi r^3$, where r is the radius.

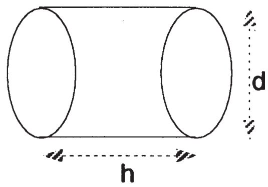
Find the volume of a sphere of (a) radius 8.5 cm (b) diameter 30 cm

13. The above formula can be transposed as $r = \sqrt[3]{\frac{3V}{4\pi}}$ which calculates the radius *r* for a given volume V.

Find the radius of a sphere whose volume is (a) 500 cm^3 (b) 1 litre

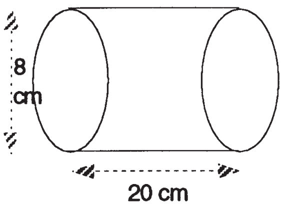
14. The formula for the surface area ($units^2$) of a closed cylinder is

$$A = \frac{\pi d}{2}(d + 2h)$$

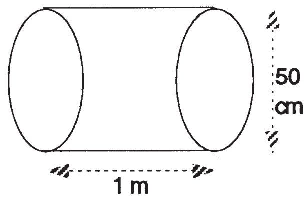


Find the surface area of each of these closed cylinders.

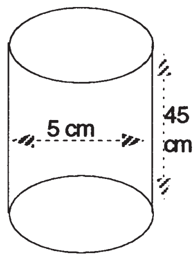
(a)



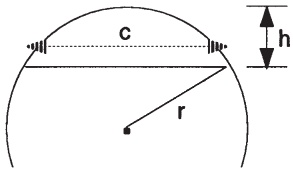
(b)



(c)



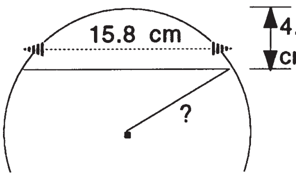
15. The radius (*r*) of a circle, the length (*c*) of a chord and the height (*h*) of a segment are linked together by these three formulae:



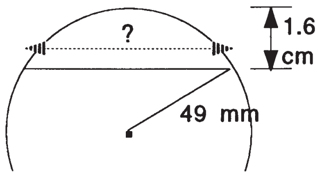
$$r = \frac{c^2 + 4h^2}{8h}, \qquad c = \sqrt{4h(2r - h)}, \qquad h = r - \frac{1}{2}\sqrt{4r^2 - c^2}$$

Apply the appropriate formula to each of these diagrams to find the missing quantity. NB The units must be consistent!

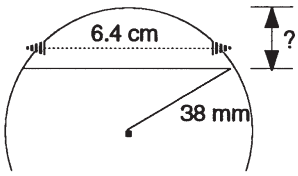
(a)



(b)



(c)



Answers

?7.1: Answers

1. 236,000
2. 5,900
3. 605,000,000
4. 418,300
5. 718,700,000
6. 41,903 (whole no.)
7. 2.4×10^{3}
8. 2.1536×10^{6}
9. 8.197×10^{2}
10. 5.38×10^{4}
11. 6.04×10^{2}
12. 2.14×10^{5}

?7.2: Answers

1. 0.000237
2. 0.00000528
3. 0.00609
4. 0.00499
5. 0.0000306
6. 0.0513
7. 7.25×10^{-3}
8. 8.53×10^{-2}
9. 7.65×10^{-4}
10. 4.34×10^{4}
11. 0.0615
12. 7.65×10^{-5}
13. 0.000831
14. 4.51×10^{5}
15. 8,200,000
16. 0.0000079
17. 4.27×10^{6}
18. 8.32×10^{-7}

? 7.3: Answers

1. 7.5×10^{-1}
2. 5.04×10^{2}
3. 1.95×10^{-5}
4. 1.53×10^{3}
5. 1.789×10^{1}
6. 1.218×10^{-9}
7. 2.921×10^{-11}
8. $2.91 \times 10^{0} = 2.91$
9. (a) $365 \times 24 \times 60 \times 60 = 31{,}536{,}000 = 3.15 \times 10^{7}$ seconds
   (b) $3 \times 10^{8} \times$ previous answer $\div 1{,}000 = 9.45 \times 10^{12}$ km
   (c) answer (b) $\times\ 4.8 = 4.536 \times 10^{13}$ km
   (d) $\dfrac{D}{S} = \dfrac{\text{answer(c) km}}{5 \times 10^{4}\ \text{km per hour}} = 9.072 \times 10^{8}$ hours $= 103{,}560$ years!
10. $2.3 \times 10^{3} \times 24 \times 365 = 2.015 \times 10^{7}$ km
11. $D \div S = 1.148 \times 10^{12}$ hours $= 1.311 \times 10^{8}$ years
12. (a) $2.4 \times 10^{3}\ \text{m/s} = \dfrac{2.4 \times 10^{3} \times 60 \times 60}{1{,}000}\ \text{km/h} = 8.640 \times 10^{3}\ \text{km/h}$
    (b) $5.0 \times 10^{5}\ \text{km/h} = \dfrac{5.0 \times 10^{5} \times 1{,}000}{60 \times 60}\ \text{m/s} = 1.389 \times 10^{5}\ \text{m/s}$
13. 50×10^{12} km in a year $= 1.585 \times 10^{9}$ m/s
14. 1.167×10^{22} atoms
15. 2.894×10^{3} days
16. 3.3×10^{24} planets

?7.4: Answers

NOTE Especially in questions 11 onwards, I have always used the π key rather than 3.14 when calculating formulae such as these. Further, I have given these answers correct to rather more decimal places than perhaps I should have.

Many conventions exist about the 'accuracy' of answers such as these. One convention is to leave the answer no more accurate than the least accurate number given in the question, in which case the answer to, say, 14(a) should be 603 and not 603.186.

Another convention is to leave answers accurate to one place more than the numbers in the question.

I have given answers more fully so that you won't get confused between what your calculator says and how a severely rounded off answer might appear in print.

1. 234.78
2. 5.439
3. 10.300
4. 14.186
5. 13.840
6. 1.678
7. 133.084
8. 388.298
9. 2.578
10. 0.254
11. 62.955 cm^3
12. (a) 2572.441 cm^3
    (b) 14137.167 cm^3
13. (a) 4.924 cm
    (b) 6.204 cm
14. (a) 603.186 cm^2
    (b) 19634.954 cm^2 or 1.963 m^2
    (c) 746.128 cm^2
15. (a) 9.407 cm
    (b) 7.244 cm
    (c) 1.751 cm

TUTOR ASSIGNMENT

Here are two questions from each of the sections of Outcome 3. Choose your questions. Don't forget to show all your working when you send in the solutions. An answer on its own is no use if it is wrong, because your tutor won't be able to help you. Only by seeing your working will the tutor know where any problems lie – this might apply even when you get the answer correct.

Applications in Social Arithmetic

T1

A person earns £53,200 per year, and pays 6% of this in superannuation, which is treated as an allowance for tax purposes. There is also the normal tax allowance of £4,250. Tax is paid on taxable income at the following rates:

First £2,000 at 20%
Basic rate thereafter up to taxable income of £40,000 at 25%
Higher tax rate on taxable income over £40,000 at 35%.
Calculate how much is left annually after all these deductions.

T2

Jenny is going on holiday to France and Germany. She changes £750 into francs and the same amount into marks when the exchange rate is £1 = 10.50 francs and £1 = 3.05 marks.

She goes to France first, spends 6,250 francs there, and changes her remaining francs into marks, the rates having now changed to £1 = 10.23 francs = 2.97 marks. She continues her holiday in Germany, and spends a total of 2,690 marks. After the holiday, she converts her remaining money back to sterling, the rate having changed again to £1 = 2.84 marks.

Assuming no problems with coinage, and no commission to pay either, how much in £s does she bring back?

More Complex Arithmetical Skills

T3

(a) A widescreen TV costs £759.99 but can be secured by paying a deposit of 20% of the cash price followed by monthly payments over 3 years of £25.99. What is the total price paid for the TV using these credit terms? Express the extra amount you are paying over and above the cash price as a percentage.

(b) A hi-fi costs £800 but can be bought on credit on the following terms:
either: pay 30% in interest and pay the whole lot off in monthly payments over 2 years
or: pay 40% in interest and pay the whole lot off in monthly payments over 3 years.
A customer can afford to pay no more than £35 per month. Which option should the customer take? Or are they both out of reach?

T4

A car depreciates in value by 20% in its first year, then every year by 15% of its value at the start of that year. It is bought at a value of £15,800, and the owner wants to sell it when its value falls below half of its original value. When should it be sold?

Ratio and Proportion

T5

A famine relief agency has 5,000 refugees in a camp with 200 crates of powdered milk which is estimated to last for 30 days. However there is a sudden influx of a further 10,000 refugees and the next scheduled air drop is in 2 weeks' time. Will the milk supplies last out? When must the air drop take place?

T6

Four pumps each working 10 hours a day can raise a total of half a million litres of water from a flooded mine in 8 days. At least that's the theory. In practice, after 2 days, only 50,000 litres of water have been raised and the pump which hasn't been pulling its weight packs in. For how many hours per day must the remaining pumps work to clear the rest of the water out, assuming it doesn't rain in the meantime?

Index Numbers

T7

This table shows the index of subscriptions to a golf club. The base year is 1990.

Year	1998	1999	2000	2001
Index	150	165	190	200

(a) An adult subscription in 1990 was £126. What was it in 1998?
(b) A child's subscription in 1999 was £36.30. What was it in the year 2000?
(c) Change the index for each year so as to make 1998 the base year.

T8

This table shows the prices and quantities of a company's exports of chocolate bars over two years. Calculate Paasche's Price Index, taking the earlier year as the base year.

	1998 thousands of items	1998 price per thousand items	2000 thousands of items	2000 price per thousand items
Choconut	23.5	£450	35.6	£400
Hazelbar	54.7	£390	50.0	£460
Crunchochoc	19.2	£120	21.4	£150

Statistical Calculations

T9

Calculate the mean and standard deviation for these mid-day temperatures in June:

Temp (°C)	15-18	19-22	23-26	27-30
No. of days	10	11	7	2

T10

Calculate the mean and standard deviation of the diameters of these washers produced by an errant machine in a factory.

Diameter (mm)	15.0-15.4	15.4-15.8	15.8-16.2	16.2-16.6	16.6-17.0
No. of washers	4	15	48	24	9

Formulae and Scientific Notation

T11

(a) Convert 3.2×10^4 km per hour into (i) km per year (ii) m per second

(b) Convert 8.4×10^{12} km per year into (i) km per hour (ii) m per second

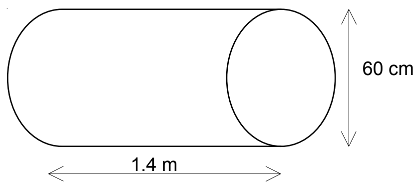
(c) A rocket flies at 2.7×10^3 m per second for 10 years. How many kilometres has it flown?

(d) How many atoms, each 2.3×10^{-14} metres in diameter, will fit into the width of this page?

T12

(a) The surface area of a cylinder is given by the formula $A = 2\pi r(r + h)$ where r is the radius and h is the height.

Find the surface area of the cylinder illustrated here. Remember that the units must be consistent.



(b) It doesn't matter what these formulae are about, just replace the appropriate letters by numbers and evaluate them:

(i) $A = P\left(1+\frac{r}{100}\right)^n$ when $P = 450$, $r = 8.5$ and $n = 4$

(ii) $f = \left(\frac{m_1 - m_2}{m_1 + m_2}\right)g$ when $m_1 = 45$, $m_2 = 23$ and $g = 9.81$

(iii) $I = \frac{mnE}{mR + nr}$ when $m = 10$, $n = 4$, $E = 120$, $R = 15$ and $r = 12$