A CURRICULUM FOR EXCELLENCE
REVIEW OF RESEARCH LITERATURE
MATHEMATICS

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We report our review of research literature in the learning and teaching of mathematics. We mainly searched articles in highly respected journals such as *Educational Studies in Mathematics*, *For the Learning of Mathematics* and *Comparative Education*. We also referred to recent volumes of *Research in Mathematics Education*, the annual publication by the British Society for Research into Learning of Mathematics (BSRLM), the UK's leading society in mathematics education research. The reviews are categorised as follows: Mathematics education as a systemic-revolutionary design science (1 article); The teaching and learning of numbers and algebra (5 articles); The use of technology and students' progress in mathematics (3 articles); Learning from other countries' mathematics education (3 articles); Teachers understandings of young children's learning in mathematics (2 articles).

Mathematics education as a systemic-revolutionary design science

Wittmann, E.: 2000, Developing Mathematics Education in a Systemic Process, Fujita, H. *et al* (Eds.): 2004, *Proceedings of the ninth International Congress on Mathematical Education*, Kluwer Academic Press, pp. 73-90.

It is a key aim of mathematics education research to improve the current situation in the learning and teaching of mathematics, in particular we consider how to encourage students' sound views about mathematics itself and mathematics learning, and to develop their mathematical thinking. One problem we often face is, however, that researchers often ignore the practical implications of their findings, and teachers often think that research findings would have little impact on classroom practice. Bridging the gap between theory and practice is fundamental and underpins improvement. A leading mathematics educator Wittmann (Germany) proposed a systemic-revolutionary view of mathematics education research which researchers and teachers can work collaboratively. (Wittmann, 1995, pp. 364-5). In a working model of research, a Substantial Learning Environment (SLE), which has flexibility and rich mathematical content (ibid, pp. 365-6), is placed at the very centre, and researchers (and teacher trainers), teachers, and children are involved in designing, implementing and researching SLEs.

In his Plenary lecture at the ninth International Congress on Mathematical Education (Tokyo, 2000), Wittmann further extended the discussions about the systemic-evolutionary view on mathematics education research. In this theoretical and philosophical paper, he fist referred to the points of view of modern systems and management theory from Descartes (Philosopher), Hilbert (Mathematician) and Comenius (Educator). According to Wittmann, the suggestions made by these prominent scholars are examples of ‘the non-systemic tradition of teaching and learning which must be fully recognized in order to be overcome’ (Wittmann, 2000, p. 77). He concluded as follows; ‘They reflect the self-concept of individuals who perceive themselves as standing on a higher level and as equipped with the capacity to bringing this field under control. The Swiss management theorist Malik has called this attitude the “mechanistic-technomorph approach to the management of complexity’ (ibid, p. 79). A systemic-evolutionary approach, on the other hand, is a new paradigm ‘which is based on the fact that biological and social organisms are far too complex in order to allow for a “mechanistic-technomorph” description and control from outside’ (ibid, p. 79). That is to say, phenomena in the learning and teaching of mathematics are highly complex and therefore it is impossible to grasp them within the “mechanistic-technomorph” paradigm, and it is necessary to employ a systemic-evolutionary approach, i.e. “Variety can only be absorbed by variety” (Ross Ashby, quoted by Wittmann, p. 76).

The latter part of this paper discusses the implication of this approach for mathematics education and challenges the traditional view of mathematics education: ‘Learning unfolds best if the spontaneous powers of all involved are brought to bear and encouraged, and if autonomy and self-responsibility are developed’; ‘The traditional borderline between the researcher on one side and the teacher on the other side has to be abandoned. Research has to build upon the spontaneous powers of teachers in the same way as teaching has to build upon the spontaneous powers of students’; ‘At all levels the traditional hierarchies have to be transformed into networks of co-operation and mutual support’ (ibid, p. 80).

Several examples of the SLE, the core of the systemic-evolutionary research in mathematics education, are then introduced. Here, a SLE is presented as a task which provide us with a ‘holistic’ situation in the learning and teaching of mathematics. For example (ibid, pp. 83-4):

3   4   3x6 + 4x7 = 18 + 28 = 46   4   5   4x8 + 5x9 = 32 + 45 = 77
| x | | x |
6   7   3x7 + 4x6 = 21 + 24 = 45   8   9   4x9 + 5x8 = 36 + 40 = 76

The example above is simple but rich mathematical and pedagogical issues are included. Wittmann described this task as follows; ‘After sufficiently many calculations with numbers chosen by the children themselves a pattern is recognized: the result obtained “top down” seems always bigger than the result obtained “crosswise”’ (ibid, p. 84). A ‘holistic’ situation becomes clear here, i.e. not only can children practice basic skills in calculations but also some mathematics thinking will be involved. As we can expect, this task can be easily extended for use in higher level of the learning and teaching mathematics (and this is one of the properties of the SLEs), e.g. the use of algebra, deductive reasoning in numbers, etc. In mathematics education research within the systemic-evolutionary paradigm, researchers and teachers collaboratively design SLEs, implement them in a classroom, and consider how we could improve the current situation in the learning and teaching of mathematics. SLEs can be also used in teacher education. Trainee teachers will experience mathematics through undertaking and designing the SLEs (ibid, p. 87).

Wittmann’s paper provides us with useful insight how we could undertake future research projects in the learning and teaching of mathematics in Scotland where we feel that much more research is required in order to improve the current situation. A project MATHE2000, which is based on the systemic-evolutionary approach of Wittmann, has been successful in Germany, and the new textbook *Das* Zahlenbuch (Wittmann and Muller) is becoming very popular. So, why cannot we do a similar thing in Scotland? This approach not only provides us an effective working model in mathematics education research, but also shows how we should design a research project, and what tasks we should use. It would be interesting to undertake research in which we design SLEs, implement them, and examine critically what happened in classrooms. Such research might contribute not only to bridge the gap between practice and theory, but also to design a curriculum which stimulates children and provides them with rich mathematical activities.

The teaching and learning of numbers and algebra

The standard of numeracy has been a central issue in the UK, and The National Numeracy Strategy (NNS, DfEE, 1999), is one of the biggest developments in recent years in England. The key features of the NNS are: an increased emphasis on number and on calculation, especially mental calculation; a three-part template for daily mathematics lessons; detailed planning using a suggested week-by-week framework of objectives; and a systematic standardised national training programme (Brown, 2002, pp. 15-6). Although the NNS is not being implemented in Scotland, the issues raised by the introduction of this new framework are related to Scottish mathematics education, e.g. students’ standard in calculation, mental mathematics, teachers and teaching, etc. In this section, we mainly review the research papers, which reported recent development of the learning and teaching of mathematics, mainly in number and calculation after the introduction of the NNS.

The impact of the National Numeracy Strategy - the Leverhulme Numeracy Research Programme

Brown, M.: 2002, Researching primary numeracy, In Cockburn, A. D. and Nardi E. (Eds), *Proceedings of the 26th Conference of the International Group for the Psychology of Mathematics Education, Vol. 1,* pp. 15-30.

A research team at King's College, University of London, undertook *the Leverhulme Numeracy Research Programme*, which was originally to investigate issues in numeracy in the UK since 1997. This project then also tracked the impact of the NNS after the introduction of this strategy in 1999. Brown (2002) reported the part of their findings at the 26th Conference of the International Group for the Psychology of Mathematics Education (PME) held at Norwich in July 2002. The programme combined a large-scale survey in a core project with case-study data in five 'focus projects' (ibid, p. 17). The core project aimed to obtain data 'to inform knowledge about the progression in pupils' learning of numeracy throughout the primary years, and to assess relative contributions to gains in numeracy of the different factors to be investigated in the programme (ibid, p. 17). The five focus projects are as follows: Focus 1) Case-studies of pupil progress; Focus 2) Teachers' conceptions and practices and pupils' learning; Focus 3) Whole school action on numeracy; Focus 4) School and community numeracies; Focus 5) Primary Cognitive Acceleration in Mathematics Education (CAME) (ibid, pp. 17-9).

Various results are reported in this paper. We comment on the results from the pupil progress. Brown summarised the results as follows: 'a) Lessons aimed at accelerating the cognitive level of children's mathematical thinking appear to show some generic results, but the result in terms of national assessment levels are more ambivalent; b) The proportion of pupils who can answer a specific question increases with age approximately following a cumulative normal distribution with variations from this relating to the curriculum and testing regime. Changes in the curriculum as a result of the National Strategy have had a significant effect on attainment in some areas; c) Over several years most children remain at roughly the same percentile of attainment, although with some oscillation. A few however gradually change their relative positions, The progress of some children appears to be held up because of some fundamental conceptual gaps' (ibid, pp. 2-23):.

Detailed examinations of data suggest that there are some positive effects (six out of 65 items) of the introduction of NNS on children's progress (ibid, p. 21). They considered that this progress was due to curriculum change rather than changes in generic pedagogy, in particular 'increased early emphasis on the number line, inverse operations and horizontal recording' (ibid, p. 21). However, it is interesting to note that 'there is no evidence of significant improvement in multiplication facts' (ibid, p.

21) from 1997 to 2002 which was a key aim of the NNS. Whilst efforts were made to encourage students' mathematical thinking (CAME), there was no significant effect (ibid, p. 20). Case studies also showed some oscillation in the progress of children. Considering these results, it is still too early to say that the NNS and other efforts to improve numeracy have significant improvement in the pupils' attainment of numeracy. Brown's view is that 'the progress of individual children shows many variations and appears to depend on many factors' (ibid, p. 23), and we feel that it is necessary to continue research in these aspects.

The impact of the National Numeracy Strategy - The Nuffield Year 4 project

Millett, A., Brown, M. and Askew, M.: 2004, The impact of the National Numeracy Strategy in Year 4 (I): attainment and learning, In McNamara, O. and Barwell, R. (eds.). *Research in mathematics education. vol. 6*, BSRLM , pp. 175-90.

Millett, A., Askew, M., and Brown, M.: 2004, The impact of the National Numeracy Strategy in Year 4 (II):teaching, In McNamara, O. and Barwell, R. (eds.). *Research in mathematics education. vol. 6*, BSRLM, pp. 191-205.

The latest findings of the *Leverhulme Numeracy Research Programme* are further reported by Millett, Brown and Askew in Chapters 10 and 11 in *Research in Mathematics Education vol. 6* in 2004. The Nuffield Year 4 project is a part of the Leverhulme Programme, and chapter 10 reports the findings of the changes in attainment using specially designed test of numearcy of Year 4 children (Millett, Brown and Askew, p. 175). A significantly large set of data, mainly tests 'to assess conceptual understanding and cognitively based skills in numeracy' (ibid, p. 177), was collected twice, i.e. in 1997/8 (before the NNS) and 2001/2 (after the NNS). Thirty-five schools were selected with the careful consideration of the following five variables, i.e. size, religious affiliation, socio-economic status of intake, attainment in national mathematics test and mathematical value added, were considered, and therefore most types of schools were included (ibid, p. 176).

Generally, evidence suggests that the NNS was effective to help children's progress in mathematics, but the effect was very small: 'This difference [3%] between 1997/8 and 2001/2 is highly statistically significant, although probably disappointing to those who expected that the NNS would cause a large increase in attainment' (ibid, pp. 177-8). It is still encouraging that the NNS had positive effects on items relating to numbers and number system and place value (ibid, p. 186). Also, 'interview responses in 2002 and 2003 indicate that the area of greatest improvement noted by schools and teachers was in pupil's mental mathematics' (ibid, p. 184). In particular, it is very interesting that the NNS has benefited boys and pupils in the middle range of attainment more than other group, mainly because of more whole class work (pp. 177-185). However, divisions and word problems are remained as the weakest areas of attainment (ibid, p. 185). Thus, they again concluded that 'the major impact on pupils' attainment and learning would appear to be due to changes in the curriculum brought in by the Strategy, with additional impact from increased length and/or frequency of mathematics lessons' (ibid, p. 189).

They report also their latest finding on the impact of the NNS on teacher and teaching in chapter 11 of *Research in Mathematics Education vol. 6*. Again, a significantly large set of data, mainly from observation of lessons, questionnaires and interviews to teachers, was collected in two phases (Millett, Askew and Brown, pp. 191-2). Then they carefully selected data from 25 lessons and 25 interview from 1997/8, 30 lessons and 29 interview from 2001/2, and 10 interview in 2003. Through analysing data, they then identified the following seven key features of the NNS (ibid, p. 193-5): Reviewing and adjusting planning and teaching using the Framework; A daily, dedicated [three-part] mathematics lesson of 45-60 minutes for all children; Greater emphasis on whole-class teaching, promoting participation; Controlled, manageable differentiation; An emphasis on mental calculation, with interactive oral and mental work; Regular activities and exercises for children to do out of class and at home; Advice, support, and training to help schools develop their practice. For example: 'Teachers in 2002 felt that more mental recall and calculation were involved in their lessons and considered it more important for pupils to be able to carry out calculation in their heads (ibid, p. 194). Comparing lessons before and after the NNS, their main findings is as follows; Comparing lessons before and after the introduction of the NNS, three main changes can be identified: Structure of the lessons; Objective or activity driven; Questions and explanations (ibid, p. 196). In summary, the findings of the impact on teacher and teaching of the NNS is quite positive.

It is particularly encouraging to see that the lessons are now more structured, mental calculations are emphasised, professional development for teachers is supported by the NNS, and that these are also concerns in Scottish mathematics education.

Qualitative study in Mental calculation strategy

Murphy, C.: 2004, How do children come to use a taught mental calculation strategy?, *Educational Studies in Mathematics*, **56**, pp. 3–18.

As we have seen, King's college research team found that 'the area of greatest improvement noted by schools and teachers was in pupil's mental mathematics' (Millett, Brown and Askew, p. 184). This is particularly interesting since this is also an important issue in Scottish mathematics education. Here, we extend our review on this topic by referring to a research paper that used a qualitative approach. The paper chosen here was recently published in *Educational Studies in Mathematics*, one of the most important journals in mathematics education.

Murphy explored in more detail how children interpret mental calculation strategies taught through whole class instruction. In particular, the compensation strategy was chosen. This strategy is described as a sophisticated method, for example, ‘12+9 would become 12+10-1. In this way a known fact, such as 12+10, is used to derive an unknown fact such as 12+9’ (Murphy, 2004, p. 5). The research design was interesting. First, the author chose three children aged 8 to 9 who were considered to be average ability in mathematics, and who demonstrated contrasting spontaneous calculation approaches, through diagnostic interviews. Then these children attended in a session to learn the compensation strategy. Finally, post-teaching diagnostic interviews were carried out a week later (ibid, pp. 5-7). Findings of this research is very interesting, but it might be rather disappointing for those who strongly believes the effectiveness of direct teaching of mental mathematics, i.e. ‘The children did not use the strategy consistently as taught and suggest contrasting interpretations’ (ibid, p. 12).

The paper then extended how we could interpret these findings. The author particularly referred to constructivism perspectives to support children’s idiosyncratic use of taught strategies ‘as it recognises the development of arithmetic from personal knowledge’ (ibid, p. 14). The author argued while ‘children’s use may be influenced and shaped by the classroom discourse’, it is also important to recognise that ‘it may also be shaped by their previous experiences and knowledge’ (ibid, p. 15). In particular, the strategy used in this study, compensation, relies on ‘knowledge of the addition or subtraction of a multiple of ten to a multi-digit number’, and therefore, .some children who have not had (or developed) this knowledge before the teaching session were unable to use the taught strategy (ibid, p. 15). For example, one girl Caroline could not use the strategy taught through the session, and ‘she appeared unable to add on a multiple of ten without counting on in ones’ (ibid, p. 11). Evidence from pre-interview showed that ‘she had used a counting procedure to add a multiple of ten’ (ibid, p. 13). As the author referred to, Gray (1997) already showed that low attainment children many not make the connections between their old and new knowledge, and Caroline’s attempts maybe explained by this.

Although the sample size was small (3 children), the results support the findings from previous large scale studies. We feel that this study raises an important issue in the learning and teaching of mental mathematics. We do want our children to develop their mental mathematics skills, and we do keep trying to teach these skills. We often tend to assume that the direct instruction spontaneously develop children’s mental calculation strategies. However, evidence in this paper suggests that this is not straightforward, but it may depend on the personal mathematical knowledge of children. In future, whilst we should keep teaching mental mathematics skills, we also have to carefully monitor the development of children’s mathematical knowledge to make our instruction more effective.

Children’s Algebraic thinking

Houssart, J. and Evens, H.: 2003, Approaching Algebra through Sequence Problems: Exploring Children’s Strategies, In Pope, S. and McNamara, O. (eds.). *Research in mathematics education. vol. 5*, BSRLM , pp. 215-241.

Algebraic thinking is regarded as a high level of mathematical thinking in numbers. In general, the teaching of algebra starts at the end of the primary school or beginning of secondary schools, but it is widely recognised that children often find difficulties in algebra. In this section we review the study by Houssart and Evens which examined 11-years-old children’s pre-algebraic thinking. The authors first discussed their theoretical framework of their study. They particularly used the ‘roots algebra as ‘”Strands or ideas which underlie algebraic thinking’ defined by Mason *et al* who identified four roots of algebra; expressing generality; rearranging and manipulating; possibilities and constraints; generalised arithmetic (Houssart and Evens, 2003, p. 198; Mason *et al*, 1985). Data was collected from 451 test papers of KS2 National Curriculum level 3-5 from 2001. The mathematical topic chosen was ‘sequences and patterns’ which is appropriate to introduce algebraic thinking to children. The task, finding the relationship between the number of circles and squares used in this study (fig. 1), is carefully designed, and this design made this study different from previous studies in this area (Houssart and Evens, 2003, p. 200).

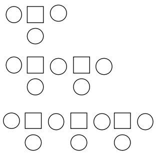


fig. 1 The question ‘Squares and circles’

The overall findings were as follows: Correct answer 37% (168 pupils); No response, 14% (65 pupils); Incorrect answer, 48% (48 pupils). Evidence from the correct answers showed that there was a wide variety of solutions to the task, e.g. use of .relationship between the numbers of circles and squares, drawing diagrams, use of tables, etc. They concluded however that it is difficult to determine the best strategy’ to solve the task (ibid, p. 210), because children’s solutions have both strengths and weaknesses. (ibid, p. 210). Incorrect answers also provided us with rich information about children’s algebraic thinking, they found common errors, e.g. assuming ‘one square to every three circles, arriving at an answer of 8, 9 or something in between’ (ibid, p. 208). In general, this study provides us with rich information how children solve patterns and sequences in mathematics, although we feel that it might be more interesting if the authors could extend their analysis to include why children adopted various strategies to solve the tasks, for example an approach that considers prior knowledge and uses this to inform the findings.

A suggestion fro teachers from this study is that they should take opportunities to work alongside children while drawing diagrams or building models in order to observe whether this informs understanding of the structure. …' (ibid, pp. 212-3). The findings and implications showed in this paper are indeed useful in Scotland where pattern and sequences are important as well as challenging in the learning and teaching of mathematics. It would be interesting to carry a similar research project in Scottish context, and compare results with the findings of Houssart and Evens. It is also useful to consider current issues of subject knowledge of trainee teachers. We, as teacher trainers, often use pattern and sequences tasks as one of problem solving activities. From our informal observations we feel that this is one of the most difficult tasks for trainee teachers in primary schools. This study provides us with a systematic framework for a research design, and information how we should analyse our students' mathematical behaviour in algebra.

The use of technology and students' progress in mathematics

The use of technology such as computers has been recognised as important in today's learning and teaching of mathematics. In Scotland, National Guidelines *Mathematic 5-14* states as follows: 'Microcomputer have an important role to play in learning mathematics. They can motivate learning, support different kinds of learning and be a vital too when using or applying mathematics' (SOED, 1991, p. 86). This is a need however for appropriate research in order to examine the effects of ICT on learning including what tasks we should design and how students develop mathematical knowledge through using computers. In this section, we review three papers in which the researchers examined how computers influence students' mathematical knowledge, in particular focusing on mathematical reasoning.

Dynamic geometry learning environments and students' reasoning

Jones, K.: 2000, Providing a Foundation for Deductive Reasoning: Students" Interpretations when Using Dynamic Geometry Software and Their Evolving Mathematical Explanations, *Educational Studies in Mathematics*, vol. **44**, pp. 55-85.

Sinclair, M.: 2003, Some Implications of the Results of a Case Study for the Design of Pre-constructed, Dynamic Geometry Sketches and Accompanying Materials, *Educational Studies in Mathematics*, **52**, 289-317.

In 2001, the UK Royal Society published a report on the teaching and learning of geometry. Interestingly this report emphases the teaching of geometry as teaching of deductive reasoning which almost disappeared from the UK curricula in the 1960s (Royal Society; 2001, p. 9). The reports suggest that 'the most significant contribution to improvements in geometry teaching will be made by the development of good models of pedagogy, supported by carefully designed activities and resources, which are disseminated effectively and coherently to and by teachers (ibid., p. 19).

Dynamic Geometry Programmes such as *Cabri Geometre* or *Geometer's Sketchpad* can be a new pedagogical tool in the learning and teaching of geometry. In a Dynamic Geometry Environment (DGE), we can interact geometrical concepts in interesting ways; we easily and precisely take points and draw lines and shapes on a computer screen, 'drag' points of the objects, and move them freely. This enables us to observe invariant properties and make conjectures.

Sinclair (2003) in Canada investigated 'the benefits and limitations of using pre-constructed, web-based, dynamic geometry sketches in activities related to deductive proof at the secondary school level' (pp. 289-90). The web-based tasks in elementary geometry were designed to interact geometrical figures and help the visualisation process by setting action buttons and tools 'to highlight particular figures, to toggle details on and off, and to rotate or reflect shapes so that they could be superimposed, or viewed from the same orientation' (ibid, p. 295). The latter issue was particularly considered since she found that one of difficulties in geometry is that students cannot 'see' figures properly when they attempt to solve problems.

The design of the research was a case study; 69 students, aged seventeen to eighteen, in three classes. Students worked in pairs (ibid, p. 295), and undertook the task such as 'When do the diagonals of a parallelogram right bisect one another?'. Data was collected from 'multiple sources of information - observation field notes, videotape, a student questionnaire, and interviews with teachers', and then it was analysed 'by coding, developing categories, describing relationships, and applying simple statistical test where appropriate' (ibid, p. 294). To analyse interactions between students, the author also applied the study by Towers (1999) which examined the effectiveness of three different teaching style: (a) showing and telling, (b) leading, and (c) shepherding [An extended stream of interventions directing a student towards understanding through subtle nudging, coaxing, and prompting]' (Towers, 1999 quoted by Sinclair, p. 292). Various responses of students were observed when they were engaged the web-based tasks; noticing, taking action, surprise, inviting, transformations and visual ideas, explaining, and avoiding trivial responses. In summary, a positive result was obtained from the evidence, i.e. 'the materials used in a geometry investigation involving pre-constructed sketches strongly impact the student' (ibid, p. 311). As implications for teaching, the following suggestions were proposed by the author to effectively facilitate DGEs to students; the provision of the visual stimulus, open-ended questions, support experimentation to unmask the confusion, inclusion of questions that check understanding (ibid, p. 312).

Keith Jones (2000), Southampton, also investigates how students interpret geometrical objects and make relationships when they use *Cabri Geometre*; 'This paper concentrates on how the students reason about geometrical objects and relations as they experience them through the dynamic geometry environment and how the mathematical explanations they offer evolve as they become more experienced both with geometry and with the software' (ibid, p. 58). Data is taken from a longitudinal study of 12 year old students. The classification of quadrilaterals was chosen for research tasks and teaching units for students. Students, tested in terms of van Hiele levels, worked in pairs, and undertook the following tasks in three phases: Phase 1: Preliminary experience with *Cabri Geometre*; Phase 2: Geometrical construction; Phase 3: Students were asked to undertake a series of six tasks that involved relationship between various quadrilaterals. (ibid, pp. 65-8)

From evidence of video and audio tape and researchers' field notes, Jones found that 'as students worked through the teaching unit, there was a shift in their thinking from imprecise, 'everyday' expressions, through reasoning mediated by the software environment to mathematical explanations of the geometric situation' (ibid. p. 80), and 'this latter stage, it is suggested, should help to provided a foundation on which to build further notions of deductive reasoning in mathematics' (ibid, p. 55). Thus, the DGEs do influence positively to shape students' deductive reasoning skills in geometry. Also, with regard to the mediational impact of using the dynamic geometry programme, 'the evidence from this study indicates that using dynamic geometry software does provide students with access to the world of geometrical theorems but it is access that is mediated by features of the software environment, certainly in the vital early and intermediate stages of using the software' (p. 81).

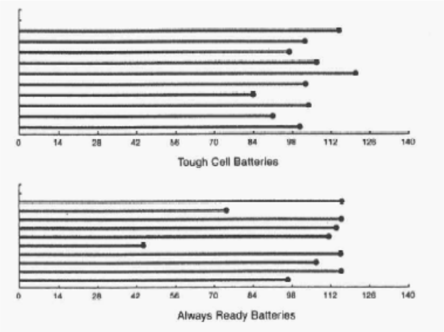
In summary, the studies reviewed above indicate that the dynamic geometry programme can be a strong tool to help students' progress in geometrical reasoning. Yet, we should not expect that students' reasoning skills would be spontaneously develop by the DGEs. Sinclair's study, in particular the recommendations above, is useful when we design DGEs for effective learning. In Scotland, Jones' study is also useful to consider how we could improve the current situation of students' understanding of relationship between quadrilaterals. The topic is currently at Level E in the *Mathematics 5-14* guideline, so it is recommended that all Scottish students are expected to understand this by the end of S2. We have not undertaken a systematic research in primary and secondary schools, we do have data from the students in our teacher training course. For example, only 57% of our students (BEd2) can answer correctly to a question 'Is a square a rectangle?' We also gave the same task as Jones used in his research to our students (BEd3), and many students could not see 'a square is a special case of a rectangle'. These imply that primary school teaches would not have the required level of understanding in geometry in Level E of *Mathematics 5-14*, and therefore children would not have enough understanding in this topic. As the studies reviewed in this section suggest, the introduction of the DGE might improve this situation, and it is an interesting area for future research in Scottish mathematics education.

The use of computer and statistical reasoning

McClain, K. et al: 2000, Supporting Students' Ways of Reasoning about Data *Learning Mathematics for a New Century*, 2000 Yearbook, pp. 174-187.

An interesting research was carried in the USA in which the researchers examined the impact of specially designed computer tools to students' statistical reasoning. The research design was an experimental approach, the authors designed two computer tools, and implemented a series of lessons in twelve weeks. The first tool was designed to provide 'a means for students to manipulate, order, partition, and otherwise organize small sets of data in a relatively routine way' (ibid, p. 176, fig. 2). The second tool was the immediate successor of the first one, and 'the endpoints of the bars that each signified a single data point in the first minitool were, in effect, collapsed down onto the axis so that a data set was now shown as collection of dots located on an axis' (ibid, p. 177). A series of lessons based on the tools in twelve weeks were implemented. Data were mainly collected by observations.

Fig. 2 The first tool (this is taken from the article, p. 179)



The episodes from classroom described in this paper clearly show how students' statistical reasoning was developed. For example, through using the first tool, students were asked to compare two separate brands of batteries, Always Ready and Tough Cell. The interactions observed were quite interesting, and the authors reported that students started to reason and justify their argument here, and the tool, the value bar, helped to encourage discussion. It was also observed that students organised, interpreted and reasoned data using the second tool as well. In the end of the experiment, evidence suggested that 'almost all the students were now able to make and understand arguments that focused on patterns in how the data were distributed' (ibid, p. 186). As an implication for teaching, this study also suggests 'the importance of middle school curricula that allow students to engage in genuine problem solving that supports the development of central mathematical concepts' (ibid. p. 186). Again, this implies the importance of designing tasks in mathematics. We are very interested in the papers by Wittmann, Brown, Sinclair, Jones, and this paper all suggest the importance of task (and curriculum in a broader sense) designing in mathematics education.

Learning from other countries' mathematics education

One thing which is suggested from the reviews above is that the design of curriculum is highly important to shape 'good' mathematical knowledge of students (see also Hoyles, 1997). To reflect our own curriculum design, it is useful to kook at various curricula in other countries. In the final section, we review research papers which adopt a comparative approach, in particular, the analysis of textbooks.

Textbooks are written materials for students both inside and outside lessons, sometimes including teaching guides for teachers. They are widely used in today's classroom. For example, Trends in International Mathematics and Science Study (hereinafter TIMSS) confirms that textbooks are still the main resources in mathematics lessons (see, for example, Foxman, 1999; Schmidt et al, 2001, p. 357; Valverde et al, 2002). In a tripartite model of educational opportunities, a curriculum is conceptualised as intended, implemented and attained curriculum. In general, the intended curriculum adopted by different countries is experienced by pupils through the textbooks that are used in their classrooms (and for homework), i.e. textbooks are potentially implemented curriculum, which mediate between intended and implemented curriculum (Valverde *et al*, 2002, p. 5.). From these reasons, it is worth researching the design and contents of textbooks.

Sutherland, R., Winter, J. and Harries, T.: 2001, A translational comparison of primary mathematics textbooks: The case of multiplication, In Morgan, C. and Jones, K. (eds.). *Research in mathematics education. vol. 3*, BSRLM , pp. 155-67.

Haggarty, L. and Pepin, B. (2002). 'An Investigation of Mathematics Textbooks and their Use in English, French and German Classrooms: who gets an opportunity to learn what?'. *British Educational Research Journal*. Vol. 28. No. 4, pp. 567-90.

Fujita, T. and Jones, K. (2003b). Interpretations of national curricula: The case of geometry in Japan and the UK, *British Educational Research Association 2003 Annual Conference*, Heriot-Watt University, Edinburgh, 11-13 September 2003.

Sutherland, Winter and Harries examined textbooks in primary schools in England, France, Hungary and Singapore. They hypothesied that 'pupils' construction of knowledge cannot be separated from the multifaceted external representations of this knowledge which envelope the learning pupil' (Sutherland, Winter and Harries; 2001, p. 155), i.e. textbooks, one such external representation, can influence and 'shape' students' mathematical knowledge. They chose the introduction of multiplication as the main mathematical topic for investigation.

As they stated, one might assume that 'mathematical notions such as multiplication are culture free' (ibid, p. 165). Their findings however challenge this assumption. Whereas all textbooks analysed in this paper introduce the multiplication as repeated addition of equal groups (ibid. p. 157), the textbooks in England showed some issues for discussion. For example, in the English textbooks, 'there is less emphasis on mathematical structure with respect to the way in which image represented on the page and the images are mainly of the object-illustrative type' (ibid. p. 160). The introduction symbols is separated from mathematical concepts in English textbooks, (ibid, p. 161). The array notion is a useful tool to extend understanding of m x n multiplication, but they also found that 'there is very little support for pupils to link the numeric representation of decomposition to the array notion' (ibid p. 161). Less use of 'real' situations are also argued by the authors (ibid, pp. 161-2). These findings imply that the way of introduction of multiplication in England might show some weaknesses to consolidate solid understanding compared to other countries, and maybe in Scotland where the design of textbooks/schemes are very similar to those in England.

Haggarty and Pepin also examined the textbooks for years 7, 8 and 9 in lower secondary schools and how they are used by teachers in classrooms in England, France, and Germany (Haggarty and Pepin; 2002, p. 567) in an article published in *British Educational Research Journal.* They particularly chose these three counties 'because the education systems of those countries were among the most influential in Europe (ibid, p. 574). The mathematical topics are 'angle' and 'directed numbers' (ibid, p. 574). Methodologically, they first analysed textbooks in the three countries, and then used the semi-structured interview to teachers' views underpinning their use of textbooks in the classroom. They also observed classroom to triangulate the data obtained through interviews (ibid, p. 574).

Like the study by Sutherland et al above, they also found some differences in the design and structures of the textbooks between the three countries. For example: 'Textbooks in England are less densely packed and contain fewer examples than textbooks in either France and Germany. Examples are almost always closed and usually follow on from a worked example almost identical to questions in the exercise. Pages are much less densely packed and there is less use of language, both technical and non-techinical.' (ibid, pp. 579-80). They also found that whereas French and German textbooks regard mathematics as a formal and structured body of pre-discovered knowledge (ibid, p. 576 & p. 586), ‘mathematics appeared to be a set of unrelated but utilitarian rules and facts’ (ibid, p. 586) in English textbooks.

Finally, we take one paper from one of us presented at the BERA2003 conference, Heriot-Watt University. In this paper, we compared textbooks in Japan and Scotland focusing on the contents in geometry. The aim is to investigate how the geometry component of the National Curriculum in Japan and the National Guidelines, is interpreted by textbook writers in those countries We analysed in accordance with the method used the study by Valverde et al (2002): division of the geometry parts of textbooks into ‘units’ and ‘blocks’; coding of each ‘block’ in terms of content, performance expectations and perspectives; identifying features of geometry in the textbooks.

We found that whereas almost all lessons in Scottish textbooks start from examples with exercises and suggested activities following, Japanese textbooks adopted various different approaches. Sometimes, for example, this Japanese textbook starts from a problem solving situation, and a narrative block which recalls some facts and theorems comes later with fewer exercises. From the national curriculum specifications in Japan and Scotland, it is expected that deductive reasoning would be very prominent in Japanese textbook, whereas consolidating facts and vocabulary, problem solving, and routine procedures would be foremost in Scottish textbook. We also discuss some influences on how knowledge of geometry would be shaped in each country. For example, the design in Scottish textbook might encourage students to use and apply the facts to various problems and show routine procedures, while justifications of various statements are not completely neglected in the textbook. Currently we are continuing our analysis on textbooks, lesson plans by teachers, etc. and considering how we could design an effective curriculum in geometry.

Teachers’ understandings and young children’s learning in mathematics

Present practices in Scottish primary maths teaching and learning are structured by the 5-14 framework together with a number of documents that have been produced by SEED and HMIE (see Maclellan, Munn and Quinn, 2002 for a summary). These documents, published subsequently to the 5-14 curriculum framework, have been influenced by a number of events in the UK and abroad;

- the development of the National Numeracy Strategy in England and Wales
- Developments in the Dutch maths curriculum stemming from the empirical curriculum development undertaken by the Freudenthal Institute
- Research on maths teaching and learning in 'Pacific rim' countries carried out in the USA during the 1980s

Empirical research has played a very small role to date in establishing the curriculum framework in the UK (see Ernst 1991) for discussion of research in relation to the National Curriculum and Brown et al 1998 for discussion of research in relation to the National Numeracy Strategy). Knowledge of three kinds of research can be relevant to the development of a maths curriculum:

1. Knowledge of research in mathematics can calibrate the school curriculum to changes in the subject itself
2. Knowledge of research into teaching practices and the teaching profession can bring to the curriculum development process an informed understanding of practitioners whose engagement with the curriculum discourse will be crucial to its success
3. Knowledge of research into the learning process itself can bring an informed understanding of children's learning in maths to the research process.

Where the primary maths curriculum is concerned, the second and third types of knowledge are most relevant and the most important pieces of recent research in these categories are as follows.

Ma, L. (1999). *Knowing and Understanding Elementary Mathematics: Teachers' Understanding of Elementary Mathematics in China and the United States.* New York: Lawrence Erlbaum Associates.

This book is based on a thesis in which Chinese and American teachers were compared in their maths teaching and understanding. The data highlighted the importance of the Chinese primary teacher's 'profound' understanding of basic number, and the impact this had on the teachers' approach to lessons. This 'profundity' was not an effect of years of education, and neither was it a result of higher learning in mathematics. It was a quality that came from the Chinese teachers' attitude towards mathematics. The implications of this research are that there are factors in teaching that prescriptive maths curricula will not affect, and that we should be looking to help primary teachers develop the 'profound' mathematical understanding that Ma describes.

Gravemeijer, K.,Cobb, P. , Bowers, J. & Whitenack, J. (2000). Symbolising and Modelling in Instructional Design. In P.Cobb, E.Yackel & K.McLain (Eds), *Symbolising and Communicating in Mathematics Classrooms*. New York: Lawrence Erlbaum.

These researchers distinguished between cognitivist and transmission models of teaching and described the mathematical practices that emerged during the course of a 3-month teaching experiment. Their description maps the way in which activity in interaction can create mathematical entities that the childen then experience as separate from their activity. Their description of this teaching experiment demonstrates the difference between the Realistic Mathematics (RM) approach and the traditional modelling approach. It also points out the important pedagogical aspect of RM – that the learning trajectory is formulated for the classroom community, not for the individual student. The implications of this research are that teachers can be helped to see the mathematics in their classroom activities in a more sophisticated way, and that they should consider the impact of their teaching efforts on both individual children and on whole classroom communities.

Summary

In summary, we identified the following points from the review of the research papers:

- From the SLE research, the task design is a crucial factor in determining effective learning. For the future, a collaboration between teachers and researchers and encouragement of teachers as researchers, in designing and implementing the SLEs would enhance the attainment of children.
- Progress and progression in learning is dependent on building on sound knowledge and understanding of prior learning of mathematics.
- Curriculum design and the appropriate timing of the content is a key factor for raising attainment.
- Direct teaching of mental strategies is effective but has to be carefully structured with good knowledge of pupil attainment.
- Algebra has been a key source of concern up to including advanced higher. Investigation into the pre-algebra activities that forms the basis of algebraic thinking and the subsequent learning and teaching of algebra would benefit progression in mathematics.
- Computers can enhance mathematical development but they have to be used in a way that challenges pupils, supports experimentation, checks understanding, is interactive and develops mathematical thinking.
- Textbooks should be used to assist learning and teaching and to support mathematical thinking that is based on appropriate tasks in mathematics, and in a variety of approaches to learning and teaching.
- Teachers' understandings both of mathematics and of pedagogy have a profound impact on the quality of pupils' learning experiences.

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